Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 1, pp. 37-66

DOI: 10.4208/aamm.OA-2023-0136
February 2026

The Nonconforming Finite Element Methods for Two Transmission Eigenvalue Problems in Inverse Scattering

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Received 11 May 2023; Accepted (in revised version) 11 November 2024

Abstract. In this paper we study the nonconforming Crouzeix-Raviart element and the enriched Crouzeix-Raviart element methods firstly for the transmission eigenvalue problem of inhomogeneous media and the modified transmission eigenvalue problem in inverse scattering. Using $\mathbb{T}$-coercivity method, we prove the convergence and the a priori error estimate of approximate eigenpair for the transmission eigenvalue problem, and based on the obtained results we prove the a priori error estimate for the modified transmission eigenvalue problem by the $\mathbb{T}$-coercivity method and Gårding inequality, and further prove that the discrete eigenvalues for the problem with metamaterial background approximate the exact eigenvalue from above. We also carry out numerical experiments to validate the theoretical findings and the efficiency of the proposed methods.

AMS subject classifications: 65N25, 65N30

Key words: Transmission eigenvalue, modified transmission eigenvalue, nonconforming finite elements, error estimate, asymptotic upper bound.

1 Introduction

The transmission eigenvalue problems (TEPs) in the inverse scattering theory have important physical and mechanical background, and the transmission eigenvalues can be used to obtain the physical properties of scattering object [1–3]. So, the numerical calculation for the TEPs has been a focus of academic circle in recent years, for example, see [4–11] and so on.

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The nonconforming Crouzeix-Raviart element (CR element) was first introduced by Crouzeix and Raviart in [12] in 1973, and the enriched Crouzeix-Raviart element (ECR element) was proposed by Hu et al. [13] and Lin et al. [14]. As for the Laplace, Stokes and Steklov eigenvalue problems, the discrete eigenvalues of the CR element and the ECR element provide the asymptotic lower bound (see [13–19]) and the guaranteed lower bound (see [20–23]) for the exact eigenvalues. In this paper, we further study the CR element and the ECR element method for the TEP and the modified transmission eigenvalue problem (mTEP). Our main work is as follows:

(1) For the TEP of inhomogeneous media (see (2.2)), Bonnet-Ben et al. [24] proposed a mixed variational formula firstly. On the basis of this formula, [6,7] researched the conforming mixed finite element method, [9] studied a finite element/holomorphic operator function method, [10] discussed a virtual element method, and [11] discussed a mixed discontinuous Galerkin method. Based on the mixed variational formula in [24], this paper aims to study the CR element and the ECR element discrete schemes for the TEP (2.2). The problem is non-selfadjoint and the sesquilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ in the weak formulation (2.5) are not coercive and indefinite, respectively. Hence its error analysis is not covered by the standard theory of non-conforming FEM of eigenvalue problem.

    The $\mathbb{T}$-coercivity method was proposed by Bonnet-Ben Dhia and Ciarlet et al. [25], which is equivalent to the inf-sup condition (see [26]). By the $\mathbb{T}$-coercivity method we prove that the discrete bilinear form $\mathcal{A}_h$ is uniformly $\mathbb{T}$-coercivity, and the discrete solution operator converges to the solution operator. Then by the spectral approximation theory we prove the a priori error estimate of the approximate eigenpairs.

(2) The mTEP (3.1) is a new eigenvalue problem arising in inverse scattering of inhomogeneous media, which was introduced in [27,28]. The properties of the eigenvalues of (3.1) depend on the choice of an artificial diffusivity parameter a that can be positive (the metamaterial case [27]) or negative (the natural case [28]). The authors in [29] discussed the spectral Galerkin method for the problem with $a>0$, and [11] discussed a mixed discontinuous Galerkin method. In this paper, based on the above mentioned work, we study the CR and ECR finite element method for the problem. We show that the discrete bilinear form $\mathcal{A}_h$ is uniformly weakly coercive if $a>0$, and uniformly weakly $\mathbb{T}$-coercive if $a<0$. Using the $\mathbb{T}$-coercive method and Gårding inequality, we prove the well-posedness of the discrete scheme and the a priori error estimate of approximate eigenpairs. In particular, when $a>0$, using the orthogonality of the CR element and the ECR element interpolations, we prove that the discrete eigenvalues approximate the exact eigenvalue from above, while the eigenvalues obtained by the conforming finite element method approximate the exact eigenvalue from below (see Remark 3.1).

(3) We carry out numerical experiments which indicates that numerical results and the-