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The Nonconforming Finite Element Methods for Two Transmission Eigenvalue Problems in Inverse Scattering

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Abstract. In this paper we study the nonconforming Crouzeix-Raviart element and the enriched Crouzeix-Raviart element methods firstly for the transmission eigenvalue problem of inhomogeneous media and the modified transmission eigenvalue problem in inverse scattering. Using $\mathbb{T}$-coercivity method, we prove the convergence and the a priori error estimate of approximate eigenpair for the transmission eigenvalue problem, and based on the obtained results we prove the a priori error estimate for the modified transmission eigenvalue problem by the $\mathbb{T}$-coercivity method and Gårding inequality, and further prove that the discrete eigenvalues for the problem with metamaterial background approximate the exact eigenvalue from above. We also carry out numerical experiments to validate the theoretical findings and the efficiency of the proposed methods.

AMS subject classifications: 65N25, 65N30

Key words: Transmission eigenvalue, modified transmission eigenvalue, nonconforming finite elements, error estimate, asymptotic upper bound.

1 Introduction

The transmission eigenvalue problems (TEPs) in the inverse scattering theory have important physical and mechanical background, and the transmission eigenvalues can be used to obtain the physical properties of scattering object [1–3]. So, the numerical calculation for the TEPs has been a focus of academic circle in recent years, for example, see [4–11] and so on.

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The nonconforming Crouzeix-Raviart element (CR element) was first introduced by Crouzeix and Raviart in [12] in 1973, and the enriched Crouzeix-Raviart element (ECR element) was proposed by Hu et al. [13] and Lin et al. [14]. As for the Laplace, Stokes and Steklov eigenvalue problems, the discrete eigenvalues of the CR element and the ECR element provide the asymptotic lower bound (see [13–19]) and the guaranteed lower bound (see [20–23]) for the exact eigenvalues. In this paper, we further study the CR element and the ECR element method for the TEP and the modified transmission eigenvalue problem (mTEP). Our main work is as follows:

(1) For the TEP of inhomogeneous media (see (2.2)), Bonnet-Ben et al. [24] proposed a mixed variational formula firstly. On the basis of this formula, [6,7] researched the conforming mixed finite element method, [9] studied a finite element/holomorphic operator function method, [10] discussed a virtual element method, and [11] discussed a mixed discontinuous Galerkin method. Based on the mixed variational formula in [24], this paper aims to study the CR element and the ECR element discrete schemes for the TEP (2.2). The problem is non-selfadjoint and the sesquilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ in the weak formulation (2.5) are not coercive and indefinite, respectively. Hence its error analysis is not covered by the standard theory of non-conforming FEM of eigenvalue problem.

    The $\mathbb{T}$-coercivity method was proposed by Bonnet-Ben Dhia and Ciarlet et al. [25], which is equivalent to the inf-sup condition (see [26]). By the $\mathbb{T}$-coercivity method we prove that the discrete bilinear form $\mathcal{A}_h$ is uniformly $\mathbb{T}$-coercivity, and the discrete solution operator converges to the solution operator. Then by the spectral approximation theory we prove the a priori error estimate of the approximate eigenpairs.

(2) The mTEP (3.1) is a new eigenvalue problem arising in inverse scattering of inhomogeneous media, which was introduced in [27,28]. The properties of the eigenvalues of (3.1) depend on the choice of an artificial diffusivity parameter a that can be positive (the metamaterial case [27]) or negative (the natural case [28]). The authors in [29] discussed the spectral Galerkin method for the problem with $a>0$, and [11] discussed a mixed discontinuous Galerkin method. In this paper, based on the above mentioned work, we study the CR and ECR finite element method for the problem. We show that the discrete bilinear form $\mathcal{A}_h$ is uniformly weakly coercive if $a>0$, and uniformly weakly $\mathbb{T}$-coercive if $a<0$. Using the $\mathbb{T}$-coercive method and Gårding inequality, we prove the well-posedness of the discrete scheme and the a priori error estimate of approximate eigenpairs. In particular, when $a>0$, using the orthogonality of the CR element and the ECR element interpolations, we prove that the discrete eigenvalues approximate the exact eigenvalue from above, while the eigenvalues obtained by the conforming finite element method approximate the exact eigenvalue from below (see Remark 3.1).

(3) We carry out numerical experiments which indicates that numerical results and the-

oretical analysis are consistent and the CR element and the ECR element methods are effective.

The rest of the paper is organized as follows. In Section 2 and the Section 3, we discuss the nonconforming CR and ECR FEMs for the TEP and the mTEP, resperctively. In Section 4, we report some numerical experiments. Throughout this paper, the letter C denotes a generic positive constant independent of the mesh size h, and the letter C without subscripts may not be the same at each occurrence. We use the symbol $c \lesssim b$ to mean that $c \le Cb$, and $c \simeq b$ to mean that $c \lesssim b$ and $b \lesssim c$.

2 The TEP and its priori error estimate

Let $\Omega \subset R^d (d=2,3)$ be a polygonal domain ($d=2$) or polyheddral domain ($d=3$). Assume that A is a real-valued $d \times d$ symmetric matrix function, $n \in L^\infty(\Omega)$ is the index of refraction, and there exists two real numbers $\gamma_1 \ge \gamma_0 > 1$ such that

$$\gamma_1 |\xi|^2 \ge \xi \cdot A\xi \ge \gamma_0 |\xi|^2, \quad \forall \xi \in R^d, \quad n(x) > \gamma_0, \quad \text{a.e. in } \Omega. \tag{2.1}$$

We consider the following TEP: find $k \in \mathbb{C}$, $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\begin{cases} \nabla \cdot A \nabla u + k^2 n(x) u = 0 & \text{in } \Omega, \\ \Delta v + k^2 v = 0 & \text{in } \Omega, \\ u - v = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu_A} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where ν is the unit outward normal to the boundary $\partial\Omega$, and $\frac{\partial u}{\partial \nu_A} = \nu \cdot (A \nabla u)$.

Define the function spaces $\mathbb{V}$ and $\mathbb{W}$ as follows:

$$\mathbb{V} = \{(u,v) \in H^1(\Omega) \times H^1(\Omega), \ u = v \ \text{ on } \ \partial\Omega\}, \quad \mathbb{W} = L^2(\Omega) \times L^2(\Omega),$$

and the associated norm is given by

$$\|(u,v)\|_{\mathbb{V}} = (\|u\|_1^2 + \|v\|_1^2)^{\frac{1}{2}} \quad \text{and} \quad \|(u,v)\|_{\mathbb{W}} = (\|u\|_0^2 + \|v\|_0^2)^{\frac{1}{2}},$$

respectively. Then $\mathbb{V} \subset \mathbb{W}$ with a compact imbedding.

We use the notation $(\cdot,\cdot)_0$ to denote the inner product in $L^2(\Omega)^d$ which is given by

$$\begin{aligned} (u,v)_0 &= \int_\Omega u \bar{v} dx & \text{for } \ d=1, \\ (\mathbf{u},\mathbf{v})_0 &= \int_\Omega \mathbf{u} \cdot \mathbf{v} dx & \text{for } \ d=2,3. \end{aligned}$$

Denote $\mathbf{U}=(u,v)$, $\Psi=(\psi,\varphi)$. We define two sesquilinear forms:

$$\mathcal{A}(U,\Psi)=(A\nabla u,\nabla\psi)_0-(\nabla v,\nabla\varphi)_0+(n(x)u,\psi)_0-(v,\varphi)_0,\qquad \forall U,\Psi\in\mathbb{V}, \tag{2.3a}$$
$$\mathcal{B}(U,\Psi)=(n(x)u,\psi)_0-(v,\varphi)_0,\qquad \forall U,\Psi\in\mathbb{W}. \tag{2.3b}$$

It is clear that

$$|\mathcal{A}(U,\Psi)|\lesssim\|U\|_{\mathbb{V}}\|\Psi\|_{\mathbb{V}},\qquad \forall U,\Psi\in\mathbb{V},$$
$$|\mathcal{B}(U,\Psi)|\lesssim\|U\|_{\mathbb{W}}\|\Psi\|_{\mathbb{W}},\qquad \forall U,\Psi\in\mathbb{W}.$$

From [7], the weak formulation of (2.2) is as follows: find $\lambda=k^2+1\in\mathbb{C}$ and $U\in\mathbb{V}$, $U\neq 0$, such that

$$\mathcal{A}(U,\Psi)=\lambda\mathcal{B}(U,\Psi),\quad \forall\Psi\in\mathbb{V}. \tag{2.4}$$

Let $F=(g,f)$, the source problem associated with (2.4) is as follows: find $U^F=(u^g,v^f)\in\mathbb{V}$, such that

$$\mathcal{A}(U^F,\Psi)=\mathcal{B}(F,\Psi),\quad \forall\Psi\in\mathbb{V}. \tag{2.5}$$

In order to analyze the eigenvalue problem (2.4), we will use the $\mathbb{T}$-coercivity method for the sesquilinear form $\mathcal{A}(\cdot,\cdot)$. Define the isomorphism operator $\mathbb{T}:\mathbb{V}\to\mathbb{V}$ by

$$\mathbb{T}\Psi=\mathbb{T}(\psi,\varphi)=(\psi,2\psi-\varphi),\quad \forall\Psi\in\mathbb{V}. \tag{2.6}$$

Note that $\mathbb{T}^2$ is an identity operator. It is easy to see that $\mathbb{T}$ is bounded on $\mathbb{V}$. Theorem 1 in [26] and Theorem 1 in [7] show that $\mathcal{A}(\cdot,\cdot)$ is $\mathbb{T}$-coercive, and satisfies the inf-sup condition, namely, there hold

$$|\mathcal{A}(\Psi,\mathbb{T}\Psi)|\gtrsim\|\Psi\|_{\mathbb{V}}^2,\qquad \forall\Psi=(\psi,\varphi)\in\mathbb{V}, \tag{2.7a}$$
$$\inf_{0\neq\Phi\in\mathbb{V}}\sup_{0\neq\Psi\in\mathbb{V}}\frac{|\mathcal{A}(\Phi,\Psi)|}{\|\Phi\|_{\mathbb{V}}\|\Psi\|_{\mathbb{V}}}\geq\mu_a, \tag{2.7b}$$

for some positive constant μ_a.

Hence we can define the solution operator $\mathbb{K}:\mathbb{W}\longrightarrow\mathbb{V}$ by

$$\mathcal{A}(\mathbb{K}F,\Psi)=\mathcal{B}(F,\Psi),\quad \forall\Psi\in\mathbb{V}, \tag{2.8}$$

and it is valid that

$$\|\mathbb{K}F\|_{\mathbb{V}}\lesssim\|F\|_{\mathbb{W}},\quad \forall F\in\mathbb{W}. \tag{2.9}$$

Then $\mathbb{K}:\mathbb{W}\longrightarrow\mathbb{V}$ is compact, and (2.5) has the equivalent operator form:

$$\lambda\mathbb{K}U=U. \tag{2.10}$$

Remark 2.1. If $U^F=(u^g,v^f)$ is the solution of the source problem (2.5) corresponding to the problem (2.4), to the best of our knowledge, because of the structure of boundary conditions, it is unknown whether the following regularity estimate holds true generally:

$$\|u^g\|_{1+r}+\|v^f\|_{1+r}\leq C(\|g\|_0+\|f\|_0). \tag{2.11}$$

However, when $A(x)=aI$ and $a\neq 1$, it can be deduced from (4.65) in [1] and [30] that (2.11) holds for $r>\frac{3}{2}$. In fact, $\omega=u^g-v^f\in H_0^1(\Omega)$ and $v\in H^1(\Omega)$ satisfy

$$a\Delta\omega=a\Delta u^g-a\Delta v^f=nu^g-ng-a(v^f-f) \qquad \text{in } \Omega, \tag{2.12a}$$
$$a\gamma\cdot\nabla\omega=a\gamma\nabla u^g-a\gamma\cdot\nabla v^f=(1-a)\gamma\cdot\nabla v^f \qquad \text{on } \partial\Omega, \tag{2.12b}$$
$$\Delta v^f+v^f=f \qquad \text{in } \Omega. \tag{2.12c}$$

Thanks to [30], for (2.12a) with $\omega|_{\partial\Omega}=0$, we get $\omega\in H^{1+r}(\Omega)$ $(r>\frac{1}{2})$ and

$$\|u^g-v^f\|_{1+r}\leq C(\|g\|_0+\|f\|_0), \tag{2.13}$$

and for (2.12c) with boundary condition (2.12b), we have that $v^f\in H^{1+r}(\Omega)$ $(r>\frac{1}{2})$ and

$$\|v^f\|_{1+r}\leq C(\|f\|_0+\|\gamma\cdot\nabla\omega\|_{r-\frac{1}{2},\partial\Omega})\leq C(\|g\|_0+\|f\|_0). \tag{2.14}$$

Combining (2.13) with (2.14), we obtain (2.11) when $A(x)=aI$ and $a\neq 1$.

Let $\pi_h=\{\kappa\}$ be a regular partition of Ω with the mesh diameter $h=\max_{\kappa\in\pi_h}h_\kappa$ where h_κ is the diameter of element κ. Let $\mathcal{E}_h=\mathcal{E}_h^i\cup\mathcal{E}_h^b$ where $\mathcal{E}_h^i$ denotes the interior faces (edges) set and $\mathcal{E}_h^b$ denotes the set of faces (edges) lying on the boundary $\partial\Omega$.

For any $e\in\mathcal{E}_h^i$, there are two simplices κ^+ and κ^- such that $e=\kappa^+\cap\kappa^-$. Let ν^+ be the unit outward normal of e pointing from κ^+ to κ^- and let $\nu^-=-\nu^+$. We define the jump $[\psi_h]=(\psi_h^+-\psi_h^-)$, where $\psi_h^+=\psi_h|_{\kappa^+}$ and $\psi_h^-=\psi_h|_{\kappa^-}$.

For notational convenience, we also define the jump on the boundary faces $e\in\mathcal{E}_h^b$ by modifying the above definitions appropriately. We use the definition of jump by understanding that $v^-=0$.

We consider the following the CR element space and the ECR element space.

$$\begin{aligned}
&\mathbb{V}_h^{CR}=\Big\{\psi\in L^2(\Omega):\psi|_\kappa\in P_1(\kappa),\ \forall\kappa\in\pi_h,\ \int_e[\psi]ds=0,\ \forall e\in\mathcal{E}_h^i\Big\},\\
&\mathbb{V}_h^{ECR}=\Big\{\psi\in L^2(\Omega):\psi|_\kappa\in P_1(\kappa)+span\Big\{\sum_{i=1}^d x_i^2\Big\},\ \forall\kappa\in\pi_h,\ \int_e[\psi]ds=0,\ \forall e\in\mathcal{E}_h^i\Big\},
\end{aligned}$$

where $P_1(\kappa)$ denotes a complete polynomial space of degree 1. Denote

$$\mathbb{V}_h=\Big\{(\psi_h,\varphi_h)\in\mathbb{V}_h^{CR}\times\mathbb{V}_h^{CR}\Big|\int_e\psi_h ds=\int_e\varphi_h ds \text{ on } \mathcal{E}_h^b\Big\}, \tag{2.15a}$$
$$\mathbb{V}_h=\Big\{(\psi_h,\varphi_h)\in\mathbb{V}_h^{ECR}\times\mathbb{V}_h^{ECR}\Big|\int_e\psi_h ds=\int_e\varphi_h ds \text{ on } \mathcal{E}_h^b\Big\}. \tag{2.15b}$$

The nonconforming finite element approximation of (2.4) is : find $U_h=(u_h,v_h)\in\mathbb{V}_h$ and $\lambda_h\in\mathbb{C}$, such that

$$\mathcal{A}_h(U_h,\Psi_h)=\lambda_h\mathcal{B}(U_h,\Psi_h),\quad \forall\Psi_h\in\mathbb{V}_h, \tag{2.16}$$

where

$$\begin{aligned}\mathcal{A}_h(U_h,\Psi_h)=\sum_{\kappa\in\pi_h}\Big(&\int_\kappa A\nabla u_h\cdot\nabla\psi_h dx-\int_\kappa\nabla v_h\cdot\nabla\varphi_h dx\\&+\int_\kappa n(x)u_h\psi_h dx-\int_\kappa v_h\varphi_h dx\Big).\end{aligned} \tag{2.17}$$

Define

$$\|\Psi_h\|_h=\sum_{\kappa\in\pi_h}(\|\psi_h\|_{1,\kappa}^2+\|\varphi_h\|_{1,\kappa}^2)^{\frac{1}{2}}.$$

For any $U=(u,v)\in\mathbb{V}$, define its interpolation $\mathbf{I}_hU=(I_hu,I_hv)$ satisfying

$$\int_e\mathbf{I}_hUds=\int_e Uds,\qquad \forall e\in\mathcal{E}_h,\quad U\in\mathbb{V}, \tag{2.18a}$$

$$\int_\kappa\mathbf{I}_hUdx=\int_\kappa Udx,\qquad \forall\kappa\in\pi_h,\quad U\in\mathbb{V}, \tag{2.18b}$$

where the CR element interpolation is defined by (2.18a) and the ECR element interpolation is defined by both (2.18a) and (2.18b). Then we have (see [16])

$$\int_\kappa\nabla(U-\mathbf{I}_hU)\cdot\nabla\Psi_h dx=0,\quad \forall\Phi_h\in\mathbb{V}_h, \tag{2.19}$$

and

$$\|\mathbf{I}_hU-U\|_1\le Ch^r\|U\|_{1+r},\quad r\in[1,2]. \tag{2.20}$$

Lemma 2.1. *There holds the $\mathbb{T}-$coercivity:*

$$|\mathcal{A}_h(\Psi_h,\mathbb{T}\Psi_h)|\gtrsim\|\Psi_h\|_h^2,\quad \forall\Psi_h=(\psi_h,\varphi_h)\in\mathbb{V}_h, \tag{2.21}$$

and the following discrete inf-sup condition is valid:

$$\inf_{0\neq\Phi_h\in\mathbb{V}_h}\sup_{0\neq\Psi_h\in\mathbb{V}_h}\frac{|\mathcal{A}_h(\Psi_h,\Phi_h)|}{\|\Phi_h\|_h\|\Psi_h\|_h}\ge C. \tag{2.22}$$

Proof. From the definition of $\mathcal{A}_h(\cdot,\cdot)$ and using Young's inequality, for $\epsilon_1,\epsilon_2<1$, we deduce

$$\begin{aligned}
|\mathcal{A}_h(\Psi_h,\mathbb{T}\Psi_h)|=&\Big|\sum_{\kappa\in\pi_h}\int_\kappa(A\nabla\psi_h\cdot\nabla\psi_h+n(x)\psi_h\psi_h-\nabla\varphi_h\cdot\nabla(2\psi_h-\varphi_h)-(2\psi_h-\varphi_h)\varphi_h)dx\Big|\\
=&\Big|\sum_{\kappa\in\pi_h}A\|\nabla\psi_h\|^2_{0,\kappa}+\sum_{\kappa\in\pi_h}n(x)\|\psi_h\|^2_{0,\kappa}+\sum_{\kappa\in\pi_h}\|\nabla\varphi_h\|^2_{0,\kappa}-2\sum_{\kappa\in\pi_h}\int_\kappa\nabla\varphi_h\cdot\nabla\psi_h dx\\
&-2\sum_{\kappa\in\pi_h}\int_\kappa\psi_h\varphi_h dx+\sum_{\kappa\in\pi_h}\|\varphi_h\|^2_{0,\kappa}\Big|\\
\geq&\sum_{\kappa\in\pi_h}\Big(\gamma_0|\psi_h|^2_{1,\kappa}+n(x)\|\psi_h\|^2_{0,\kappa}+|\varphi_h|^2_{1,\kappa}-\epsilon_1|\varphi_h|^2_{1,\kappa}-\frac{1}{\epsilon_1}|\psi_h|^2_{1,\kappa}-\epsilon_2\|\varphi_h\|^2_{0,\kappa}\\
&-\frac{1}{\epsilon_2}\|\psi_h\|^2_{0,\kappa}+\|\varphi_h\|^2_{0,\kappa}\Big)\\
\geq&\min\Big\{\gamma_0-\frac{1}{\epsilon_1},\gamma_0-\frac{1}{\epsilon_2}\Big\}\sum_{\kappa\in\pi_h}\|\psi_h\|^2_{1,\kappa}+\min\{1-\epsilon_1,1-\epsilon_2\}\sum_{\kappa\in\pi_h}\|\varphi_h\|^2_{1,\kappa}\\
\geq&C\|\Psi_h\|^2_h,
\end{aligned}$$

where $C=\min\{\gamma_0-\frac{1}{\epsilon_1},\gamma_0-\frac{1}{\epsilon_2},1-\epsilon_1,1-\epsilon_2\}$. Then (2.21) holds. Hence, there exists a positive constant C such that

$$\inf_{0\neq\Phi_h\in\mathbb{V}_h}\sup_{0\neq\Psi_h\in\mathbb{V}_h}\frac{|\mathcal{A}_h(\Psi_h,\Phi_h)|}{\|\Phi_h\|_h\|\Psi_h\|_h}\geq\inf_{0\neq\Phi_h\in\mathbb{V}_h}\frac{|\mathcal{A}_h(\Psi_h,\mathbb{T}\Psi_h)|}{\|\mathbb{T}\Psi_h\|_h\|\Psi_h\|_h}\geq C,$$

and (2.22) holds. $\square$

It is easy to know that the following continuity property holds:

$$|\mathcal{A}_h(U_h,\Psi_h)|\lesssim\|U_h\|_h\|\Psi_h\|_h,\quad\forall U_h,\Psi_h\in\mathbb{V}_h. \tag{2.23}$$

The discrete form associated with (2.5) is as follows: find $U_h^F=(u_h^g,v_h^f)\in\mathbb{V}_h$, such that

$$\mathcal{A}_h(U_h^F,\Psi_h)=\mathcal{B}(F,\Psi_h),\quad\forall\Psi_h\in\mathbb{V}_h. \tag{2.24}$$

Since (2.22) holds, from [26], we can obtain the well-posedness of (2.24). Then there is the solution operator $\mathbb{K}_h:\mathbb{W}\to\mathbb{V}_h$ satisfying

$$\mathcal{A}_h(\mathbb{K}_hF,\Psi_h)=\mathcal{B}(F,\Psi_h),\quad\forall\Psi_h\in\mathbb{V}_h. \tag{2.25}$$

It is valid that

$$\|\mathbb{K}_hF\|_h\lesssim\|F\|_{\mathbb{W}},\quad\forall F\in\mathbb{W}. \tag{2.26}$$

Then (2.16) has the equivalent operator form

$$\lambda_h\mathbb{K}_hU_h=U_h. \tag{2.27}$$

Let U^F and U_h^F be the solutions of (2.5) and (2.24), respectively. Define the consistency error term of nonconforming finite element:

$$E_h(U^F,\Psi_h)=\mathcal{A}_h(U^F,\Psi_h)-\mathcal{B}(F,\Psi_h),\quad \forall\Psi_h\in\mathbb{V}+\mathbb{V}_h. \tag{2.28}$$

Lemma 2.2. *For any $F\in\mathbb{W}$, let $U^F\in(H^{1+r}(\Omega))^d$ $(r>\frac{1}{2})$ be the solution of (2.5). It is valid that*

$$E_h(U^F,\Psi_h)\lesssim h^r\|U^F\|_{1+r}\|\Psi_h\|_h,\quad \forall\Psi_h\in\mathbb{V}+\mathbb{V}_h. \tag{2.29}$$

Proof. From (2.28), we have

$$\begin{aligned}
&E_h(U^F,\Psi_h)=\mathcal{A}_h(U^F,\Psi_h)-\mathcal{B}(F,\Psi_h)\\
=&\sum_{\kappa\in\pi_h}\int_\kappa A\nabla u^g\cdot\nabla\psi_h dx-\sum_{\kappa\in\pi_h}\int_\kappa\nabla v^f\cdot\nabla\varphi_h dx+\sum_{\kappa\in\pi_h}\int_\kappa n(x)u^g\psi_h dx\\
&-\sum_{\kappa\in\pi_h}\int_\kappa v^f\varphi_h dx-\sum_{\kappa\in\pi_h}\int_\Omega n(x)g\psi_h dx+\sum_{\kappa\in\pi_h}\int_\kappa f\varphi_h dx\\
=&\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\nu\cdot(A\nabla u^g)\psi_h ds-\sum_{\kappa\in\pi_h}\int_{\partial\kappa}\nu\cdot\nabla v^f\varphi_h ds\\
=&\frac{1}{2}\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^i}\int_e\big(\nu^+\cdot(A\nabla u^g)\psi_h|_{\kappa^+}-\nu^-\cdot(A\nabla u^g)\psi_h|_{\kappa^-}\big)ds+\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^b}\int_e\nu\cdot(A\nabla u^g)\psi_h ds\\
&-\frac{1}{2}\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^i}\int_e\nu^+\cdot\Big(\nabla v^f\varphi_h|_{\kappa^+}-\nabla v^f\varphi_h|_{\kappa^-}\Big)ds-\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^b}\int_e\nabla v^f\varphi_h\cdot\nu ds\\
=&\frac{1}{2}\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^i}\int_e\nu^+\cdot(A\nabla u^g)[\psi_h]ds+\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^b}\int_e\nu\cdot(A\nabla u^g)\psi_h ds\\
&-\frac{1}{2}\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^i}\int_e\nu^+\cdot\nabla v^f[\varphi_h]ds-\sum_{\kappa\in\pi_h}\sum_{e\in\partial\kappa\cap\mathcal{E}_h^b}\int_e\nu\cdot\nabla v^f\varphi_h ds.
\end{aligned}\tag{2.30}$$

Introduce the edge mean interpolation operator and the element mean interpolation operator as follows:

$$P_0^e f=\frac{1}{|e|}\int_e f ds,\quad R_0^e f=f-P_0^e f,\quad P_0^\kappa f=\frac{1}{|\kappa|}\int_\kappa f ds,\quad R_0^\kappa f=f-P_0^\kappa.$$

For the first term on the right hand side of (2.30), we have

$$\begin{aligned}
&\int_e\nu^+\cdot(A\nabla u^g)[\psi_h]ds\\
=&\int_e\nu^+\cdot((A\nabla u^g)-P_0^\kappa(A\nabla u^g))[\psi_h]ds+\int_e\nu^+\cdot P_0^\kappa(A\nabla u^g)[\psi_h]ds\\
=&\int_e\nu^+\cdot R_0^\kappa(A\nabla u^g)[\psi_h]ds
\end{aligned}$$

$$
\begin{aligned}
&=\int_e \nu^+\cdot R_0^\kappa(A\nabla u^g)(R_0^e([\psi_h])+P_0^e([\psi_h]))ds\\
&=\int_e \nu^+\cdot R_0^\kappa(A\nabla u^g)R_0^e([\psi_h])ds\\
&\le\Big\{\int_e (R_0^\kappa(A\nabla u^g))^2ds\Big\}^{\frac{1}{2}}\Big\{\int_e (R_0^e([\psi_h]))^2ds\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{2.31}
$$

Using the trace inequality, we can estimate the first term on the right-hand side of (2.31) as follows.

$$
\begin{aligned}
\int_e (R_0^\kappa(A\nabla u^g))^2ds &\le C\{h_\kappa^{-1}\|R_0^\kappa(A\nabla u^g))\|_{0,\kappa}^2+h_\kappa^{2r-1}|R_0^\kappa(A\nabla u^g))|_{r,\kappa}^2\}\\
&\le Ch_\kappa^{2r-1}|(\nabla u^g)|_{r,\kappa}^2\le Ch_\kappa^r|u^g|_{1+r,\kappa}^2.
\end{aligned} \tag{2.32}
$$

Since $P_0^e: L^2(e)\to P_0(e)$ is an orthogonal projection, then for any $f\in L^2(e)$, we have

$$
\|f-P_0^e f\|_{0,e}\le\|f-p_0\|_{0,e},\quad \forall p_0\in P_0(e).
$$

Similarly, using trace inequality we can deduce

$$
\begin{aligned}
\int_e (R_0^e([\psi_h]))^2ds&=\int_e([\psi_h]-P_0^e([\psi_h]))^2ds\\
&=\int_e\{(\psi_h^+-P_0^e(\psi_h^+))-(\psi_h^- -P_0^e(\psi_h^-))\}^2ds\\
&\le 2\Big\{\int_e(\psi_h^+-P_0^e(\psi_h^+))^2ds+\int_e(\psi_h^- -P_0^e(\psi_h^-))^2ds\Big\}\\
&\le 2\Big\{\int_e(\psi_h^+-P_0^{\kappa^+}(\psi_h^+))^2ds+\int_e(\psi_h^- -P_0^{\kappa^-}(\psi_h^-))^2ds\Big\}\\
&\le C\{(h_{\kappa^+}^{-1}\|R_0^{\kappa^+}(\psi_h)\|_{0,\kappa^+}^2+h_{\kappa^+}|R_0^{\kappa^+}(\psi_h)|_{1,\kappa^+}^2)\\
&\qquad+(h_{\kappa^-}^{-1}\|R_0^{\kappa^-}(\psi_h)\|_{0,\kappa^-}^2+h_{\kappa^-}|R_0^{\kappa^-}(\psi_h)|_{1,\kappa^-}^2)\}\\
&\le C\Big\{h_{\kappa^+}|\psi_h|_{1,\kappa^+}^2+h_{\kappa^-}|\psi_h|_{1,\kappa^-}^2\Big\}.
\end{aligned} \tag{2.33}
$$

Combining (2.32) and (2.33), we get

$$
\int_e \nu^+\cdot(A\nabla u^g)[\psi_h]ds\le Ch_\kappa^r|u^g|_{1+r,\kappa}|\psi_h|_{1,\kappa^+\cup\kappa^-}. \tag{2.34}
$$

Next, we use the same analytical method as (2.31) to estimate the third term on the right hand side of (2.30) and get

$$
\int_e \nu^+\cdot\nabla v^f[\varphi_h]ds\le Ch_\kappa^r|v^f|_{1+r,\kappa}|\varphi_h|_{1,\kappa^+\cup\kappa^-}. \tag{2.35}
$$

Finally, we estimate the second and the fourth terms on the right hand side of (2.30)

$$\begin{aligned}
&\int_e \nu\cdot(A\nabla u^g)\psi_h ds-\int_e \nu\cdot\nabla v^f\varphi_h ds\\
=&\int_e \nu\cdot\nabla v^f(\psi_h-\varphi_h)ds\\
=&\int_e \nu\cdot(\nabla v^f-P_0^\kappa(\nabla v^f))(\psi_h-\varphi_h)ds+\int_e \nu\cdot P_0^\kappa(\nabla v^f)(\psi_h-\varphi_h)ds\\
=&\int_e \nu\cdot R_0^\kappa(\nabla v^f)(\psi_h-\varphi_h)ds\\
=&\int_e \nu\cdot R_0^\kappa(\nabla v^f)(R_0^e(\psi_h-\varphi_h)+P_0^e(\psi_h-\varphi_h))ds\\
=&\int_e \nu\cdot R_0^\kappa(\nabla v^f)R_0^e(\psi_h-\varphi_h)ds\\
\le&\Big\{\int_e (R_0^\kappa(\nabla v^f))^2 ds\Big\}^{\frac12}\Big\{\int_e (R_0^e(\psi_h-\varphi_h))^2 ds\Big\}^{\frac12},
\end{aligned}$$

which together with the trace inequality yields

$$\begin{aligned}
\int_e (R_0^\kappa(\nabla v^f))^2 ds&\le C(h_\kappa^{-1}\|R_0^\kappa(\nabla v^f)\|_{0,\kappa}^2+h_\kappa^{2r-1}|R_0^\kappa(\nabla v^f)|_{r,\kappa}^2)\\
&\le Ch_\kappa^{2r-1}|\nabla v^f|_{r,\kappa}^2\\
&\le Ch_\kappa^{2r-1}|v^f|_{1+r,\kappa}^2,
\end{aligned}$$

and

$$\begin{aligned}
\int_e (R_0^e(\psi_h-\varphi_h))^2 ds&\le C\left(h_\kappa^{-1}\|R_0^e(\psi_h-\varphi_h)\|_{0,\kappa}^2+h_\kappa|R_0^e(\psi_h-\varphi_h)|_{1,\kappa}^2\right)\\
&\le Ch_\kappa^1|\psi_h-\varphi_h|_{1,\kappa}^2.
\end{aligned}$$

From the above two inequalities, we obtain

$$\int_e \nu\cdot(A\nabla u^g)\psi_h ds-\int_e \nu\cdot\nabla v^f\varphi_h ds\le Ch_\kappa^r|v^f|_{1+r,\kappa}|\psi_h-\varphi_h|_{1,\kappa}. \tag{2.36}$$

Substituting (2.34), (2.35) and (2.36) into (2.30), we have

$$\begin{aligned}
E_h(U^F,\Psi_h)&\le Ch^r\sum_{\kappa\in\pi_h}(|u^g|_{1+r,\kappa}|\psi_h|_{1,\kappa}+|v^f|_{1+r,\kappa}|\varphi_h|_{1,\kappa}+|v^f|_{1+r,\kappa}|\psi_h-\varphi_h|_{1,\kappa})\\
&\lesssim h^r\|U^F\|_{1+r}\|\Psi_h\|_h.
\end{aligned}$$

The proof is completed. □

Thanks to the Strang lemma (for example, see Lemma 5.1.1 in [31]), we obtain the following lemma.

Lemma 2.3. *Let U^F and U_h^F be the solutions of* (2.5) *and* (2.24)*, respectively. Then*

$$\|U^F-U_h^F\|_h \backsimeq \inf_{\Phi_h\in\mathbb{V}_h}\|U^F-\Phi_h\|_h+\sup_{Q_h\in\mathbb{V}_h}\frac{E_h(U^F,Q_h)}{\|Q_h\|_h}. \tag{2.37}$$

Proof. For any $\Phi_h\in\mathbb{V}_h$, using triangle inequality we have

$$\|U^F-U_h^F\|_h\le\|U^F-\Phi_h\|_h+\|U_h^F-\Phi_h\|_h. \tag{2.38}$$

Using Lemma 2.1, we deduce that

$$\begin{aligned}
C\|U_h^F-\Phi_h\|_h^2 &\le \mathcal{A}_h(U_h^F-\Phi_h,\mathbb{T}(U_h^F-\Phi_h))\\
&=\mathcal{A}_h(U_h^F-U^F+U^F-\Phi_h,\mathbb{T}(U_h^F-\Phi_h))\\
&=\mathcal{A}_h(U^F-U_h^F,\mathbb{T}(\Phi_h-U_h^F))+\mathcal{A}_h(U^F-\Phi_h,\mathbb{T}(U_h^F-\Phi_h))\\
&=\mathcal{A}_h(U^F,\mathbb{T}(\Phi_h-U_h^F))-\mathcal{A}_h(U_h^F,\mathbb{T}(\Phi_h-U_h^F))+\mathcal{A}_h(U^F-\Phi_h,\mathbb{T}(U_h^F-\Phi_h))\\
&=E_h(U^F,\mathbb{T}(\Phi_h-U_h^F))+M\|U_h^F-\Phi_h\|_h\|U^F-\Phi_h\|_h.
\end{aligned} \tag{2.39}$$

Dividing both sides of (2.39) by $\|U_h^F-\Phi_h\|_h$ and substituting (2.39) into (2.38), noting that the arbitrariness of Φ_h, we obtain

$$\begin{aligned}
\|U^F-U_h^F\|_h &\lesssim \|U^F-\Phi_h\|_h+\frac{E_h(U^F,\mathbb{T}(\Phi_h-U_h^F))}{\|\Phi_h-U_h^F\|_h}\\
&\lesssim \|U^F-\Phi_h\|_h+\sup_{Q_h\in\mathbb{V}_h}\frac{E_h(U^F,Q_h)}{\|Q_h\|_h}, \quad \forall\Phi_h\in\mathbb{V}_h,
\end{aligned} \tag{2.40}$$

i.e., the side $\lesssim$ in (2.37) is true. On the other hand,

$$\mathcal{A}_h(U^F-U_h^F,Q_h)\lesssim\|U^F-U_h^F\|_h\|Q_h\|_h, \quad \forall Q_h\in\mathbb{V}_h,$$

then the other side in (2.37) is obvious. □

From Lemmas 2.2 and 2.3, we obtain the following theorem.

Theorem 2.1. *Let $U^F\in(H^{1+r}(\Omega))^d$ $(r>\frac{1}{2})$ and U_h^F be the solutions of* (2.5) *and* (2.24)*, respectively, then there holds the following estimate:*

$$\|U^F-U_h^F\|_h\lesssim h^r, \tag{2.41}$$

further assume (2.11) *holds, then*

$$\|U^F-U_h^F\|_{\mathbb{W}}\lesssim h^r\|U^F-U_h^F\|_h. \tag{2.42}$$

Proof. Substituting (2.20), (2.29) into (2.37), we obtain (2.41). Let $\mathbf{I}_h^c$ be the conforming linear interpolation operator on π_h, then $\mathbf{I}_h^c\Psi\in(H^1(\Omega)\times H^1(\Omega))\cap\mathbb{V}_h$. According to (2.5) and (2.24), we have $\mathcal{A}_h(U^F-U_h^F,I_h^c\Psi)$=0, thus, from (2.29), (2.41), the interpolation error estimate and (2.11), we obtain

$$\begin{aligned}\mathcal{B}(U^F-U_h^F,F)=&\mathcal{B}(U^F-U_h^F,F)-\mathcal{A}_h(U^F-U_h^F,\Psi)+\mathcal{A}_h(U^F-U_h^F,\Psi)\\ =&-E_h(U^F-U_h^F,\Psi)+\mathcal{A}_h(U^F-U_h^F,\Psi-\mathbf{I}_h^c\Psi)\\ \lesssim& h^r\|U^F-U_h^F\|_h\|\Psi\|_{1+r}\\ \lesssim& h^r\|F\|_0\|U^F-U_h^F\|_h.\end{aligned}$$

By using Riesz representation theorem, we obtain (2.42). □

Lemma 2.4. *Assume that for any $F\in\mathbb{W}$, $U^F\in(H^{1+r}(\Omega))^2$ $(r>\frac{1}{2})$ is the solution of* (2.5)*, then $\mathbb{K}_h\to\mathbb{K}$ pointwise and $\{\mathbb{K}_h\}$ is collectively compact; further assume* (2.11) *holds, then*

$$\|\mathbb{K}_h-\mathbb{K}\|_{\mathbb{W}}\to 0,\quad h\to 0. \tag{2.43}$$

Proof. From (2.41), we have that for any given $F\in\mathbb{W}$,

$$\|(\mathbb{K}_h-\mathbb{K})F\|_{\mathbb{W}}\lesssim\|(\mathbb{K}_h-\mathbb{K})F\|_{\mathbf{h}}\to 0\quad \text{as}\ \ h\to 0,$$

which shows $\mathbb{K}_h\to\mathbb{K}$ pointwise.

From (2.26) we know the set $\cup_{h\in(0,1)}\{\mathbb{K}_hF:\|F\|_{\mathbb{W}}\le 1\}$ is bounded in the sense of norm $\|\cdot\|_h$. Then by the compact embedding (see Theorem 10.3.1 in [31]), we know that the set $\cup_{h\in(0,1)}\{\mathbb{K}_hF:\|F\|_{\mathbb{W}}\le 1\}$ is sequentially compact. Hence, $\{\mathbb{K}_h\}$ is collectively compact.

From

$$\|\mathbb{K}-\mathbb{K}_h\|_{\mathbb{W}}=\sup_{F\in\mathbb{W}}\frac{\|(\mathbb{K}-\mathbb{K}_h)F\|_{\mathbb{W}}}{\|F\|_{\mathbb{W}}},$$

(2.42) and (2.11), we get (2.43). The proof is completed. □

Suppose that λ and λ_h are the kth eigenvalue of (2.4) and (2.16), respectively. Let q and α be the algebraic multiplicity and the ascent of λ, respectively. $\lambda=\lambda_k=\lambda_{k+1}=\cdots\lambda_{k+q-1}$. Let $M(\lambda)$ be the generalized eigenfunction space of (2.4) corresponding to λ and $M_h(\lambda)$ be the direct sum of the generalized eigenfunctions corresponding to all eigenvalues of (2.16) that converge to λ. Let $\widehat{M}(\lambda)=\{\Psi|\Psi\in M(\lambda),\ \|\Psi\|_{\mathbb{W}}=1\}$,

$$\widehat{\lambda}_h=\frac{1}{q}\sum_{j=k}^{k+q-1}\lambda_{j,h}$$

denotes the arithmetic means of q discrete eigenvalues of (2.16) that converge to λ.

For two closed subspace A and B of $\mathbb{W}$, we define the gap between A and B in norm $\|\cdot\|_{\mathbb{W}}$ as follows:

$$\widehat{\Theta}(A,B)=\sup_{\Phi\in A,\|\Phi\|_{\mathbb{W}}=1}\inf_{\Psi\in B}\|\Phi-\Psi\|_{\mathbb{W}},\ \Theta(A,B)=\max\{\widehat{\Theta}(A,B),\widehat{\Theta}(B,A)\}. \tag{2.44}$$

Theorem 2.2. *Assume that $M(\lambda)\subset(H^{1+r}(\Omega))^d$ with $r\in(\frac{1}{2},1]$, then*

$$\begin{aligned}
&\Theta(M(\lambda),M_h(\lambda))\lesssim\|(\mathbb{K}-\mathbb{K}_h)|_{M(\lambda)}\|_{\mathbb{W}}, &&\text{(2.45a)}\\
&|\lambda-\widehat{\lambda}_h|\lesssim\|(\mathbb{K}-\mathbb{K}_h)|_{M(\lambda)}\|_{\mathbb{W}}, &&\text{(2.45b)}\\
&|\lambda-\lambda_{j,h}|\lesssim\|(\mathbb{K}-\mathbb{K}_h)|_{M(\lambda)}\|_{\mathbb{W}}^{\frac{1}{\alpha}}, &&\text{(2.45c)}
\end{aligned}$$

where $(\mathbb{K}-\mathbb{K}_h)|_{M(\lambda)}$ denotes the restriction of $\mathbb{K}-\mathbb{K}_h$ to $M(\lambda)$. Further, let $\alpha=1$ and assume that (2.11) *holds, then*

$$\begin{aligned}
&\|U-U_h\|_h\lesssim h^r, &&\text{(2.46a)}\\
&\|U-U_h\|_{\mathbb{W}}\lesssim h^r\|U-U_h\|_h. &&\text{(2.46b)}
\end{aligned}$$

Proof. From Lemma 2.4, we know $\mathbb{K}_h\to\mathbb{K}$ pointwise and $\{\mathbb{K}_h\}$ is collectively compact. Since $\mathbb{K}:\mathbb{W}\to\mathbb{W}$ is compact, from Theorems 1, 2 and 6 in [33], we get (2.45a), (2.45b) and (2.45c), respectively. From (2.26) and the spectral approximation theory, we deduce

$$\begin{aligned}
\|U-U_h\|_h&=\|\lambda\mathbb{K}U-\lambda_h\mathbb{K}_hU_h\|_h\\
&\le\|\lambda\mathbb{K}U-\lambda\mathbb{K}_hU\|_h+\|\lambda\mathbb{K}_hU-\lambda_h\mathbb{K}_hU_h\|_h\\
&\lesssim\|\mathbb{K}U-\mathbb{K}_hU\|_h+|\lambda-\lambda_h|+\|U-U_h\|_{\mathbb{W}}\lesssim h^r,
\end{aligned}$$

i.e., (2.46a) is valid.

From triangle inequality, (2.26) and (2.42), we obtain

$$\begin{aligned}
\big|\|U-U_h\|_h-\|\lambda\mathbb{K}U-\lambda\mathbb{K}_hU\|_h\big|&\le\|\lambda\mathbb{K}U-\lambda_h\mathbb{K}_hU_h-\lambda\mathbb{K}U+\lambda\mathbb{K}_hU\|_h\\
&=\|\mathbb{K}_h(\lambda U-\lambda_hU_h)\|_h\\
&\lesssim\|\lambda U-\lambda_hU_h\|_{\mathbb{W}}\\
&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)U\|_h. \qquad\qquad \text{(2.47)}
\end{aligned}$$

It is easy to know from (2.47) that

$$\|U-U_h\|_h\simeq\|\lambda\mathbb{K}U-\lambda\mathbb{K}_hU\|_h. \tag{2.48}$$

From the spectral approximate theory, (2.42) and (2.47), we get

$$\begin{aligned}
\|U-U_h\|_{\mathbb{W}}&\lesssim\|(\mathbb{K}_h-\mathbb{K})U\|_{\mathbb{W}}\\
&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)U\|_h\\
&\lesssim h^r\|U-U_h\|_h,
\end{aligned}$$

i.e., (2.46b) holds. $\square$

3 The mTEP and its priori error estimate

Unless otherwise stated, symbols appearing in this section that are the same as those in Section 2 have the same meaning.

We consider the mTEP: find $\lambda \in \mathbb{R}$ and $u,v \in H^1(\Omega)$, $v \neq 0$, such that

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } \Omega, \\ (-a)\Delta v + k^2 \lambda v = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = -a\dfrac{\partial v}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Here the index of refraction $n \in L^\infty(\Omega)$ is a positive real function, ν is the unit outward normal to the boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu} = \nu \cdot (\nabla u)$ and $k > 0$ is the fixed wavenumber.

We transform the problem (3.1) into the following equivalent weak form: find $U \in \mathbb{V}$, $\|kv\|_0 = 1$ and $\vartheta = \beta - \lambda \in \mathbb{R}$, such that

$$\mathcal{A}(U,\Psi) = \vartheta \mathcal{B}(U,\Psi), \quad \forall \Psi \in \mathbb{V}, \tag{3.2}$$

where

$$\mathcal{A}(U,\Psi) = \int_\Omega \nabla u \cdot \nabla \psi dx + a\int_\Omega \nabla v \cdot \nabla \varphi dx - \int_\Omega k^2 n u \psi dx + \int_\Omega k^2 \beta v \varphi dx, \tag{3.3a}$$

$$\mathcal{B}(U,\Psi) = \int_\Omega k^2 v \varphi dx. \tag{3.3b}$$

It is clear that

$$|\mathcal{A}(U,\Psi)| \leq \|U\|_{\mathbb{V}} \|\Psi\|_{\mathbb{V}}. \tag{3.4}$$

Using Gårdings inequality in [34] we get the following lemma.

Lemma 3.1. *For any $a > 0$, there exists a sufficiently large constant $M > 0$ and a constant $C_1 > 0$ such that*

$$\mathcal{A}(\Psi,\Psi) + M\|\Psi\|^2_{\mathbb{W}} \geq C_1 \|\Psi\|^2_{\mathbb{V}}, \quad \forall \Psi \in \mathbb{V}. \tag{3.5}$$

For any $-1 < a < 0$, there exists a sufficiently large constant $M > 0$ and a constant $C_2 > 0$ such that

$$\mathcal{A}(\Psi,\mathbb{T}\Psi) + M\|\Psi\|^2_{\mathbb{W}} \geq C_2 \|\Psi\|^2_{\mathbb{V}}, \quad \forall \Psi \in \mathbb{V}. \tag{3.6}$$

Proof. See the proof of Lemmas 3.1 and 3.2 in [11]. □

Let $F=(0,f)$, the source problem associated with (3.2) is as follows: find $U^F=(u^f,v^f)\in\mathbb{V}$, such that

$$\mathcal{A}(U^F,\Psi)=\mathcal{B}(F,\Psi),\quad \forall\Psi\in\mathbb{V}. \tag{3.7}$$

Consider the TEP: find $(\mathrm{k}^2,U)\in\mathbb{C}\times\mathbb{V}$ satisfying

$$\mathcal{A}(U,\Psi)=0,\quad \forall\Psi\in\mathbb{V}. \tag{3.8}$$

In this section, we always assume that the following condition holds (see page 7 in [28] or page 18 in [27]): fixed a real β (if $\vartheta<0$, choose $\beta\in(0,\gamma_0)$, and if $\vartheta>0$, choose $\beta\in(0,\infty)$) such that the wavenumber k^2 is not a transmission eigenvalue of (3.8).

Note that the above assumption implies Eq. (3.8) has only zero solution. Thus, if $\vartheta>0$, from (3.5) and Lemma 35.3 in [37] we know that $\mathcal{A}(\cdot,\cdot)$ satisfies the inf-sup condition, and if $\vartheta<0$, using the same argument as in Lemma 35.3 of [37], from (3.6) we can deduce $\mathcal{A}(\cdot,\cdot)$ satisfies the inf-sup condition. Then we define the solution operator $\mathbb{K}=(S,T):L^2(\Omega)\longrightarrow\mathbb{V}$ satisfying

$$\begin{aligned}
&\mathcal{A}(\mathbb{K}f,\Psi)=\mathcal{B}(F,\Psi),\quad \forall\Psi\in\mathbb{V},\\
&\mathbb{K}f=U^F,\\
&T:L^2(\Omega)\to\mathbb{V},\quad Tf=u^f,\\
&S:L^2(\Omega)\to\mathbb{V},\quad Sf=v^f.
\end{aligned}$$

Then (3.2) can be written as the following equivalent operator forms:

$$\vartheta Tv=v,\quad S(\vartheta v)=u,$$

and

$$\|\mathbb{K}F\|_{\mathbb{V}}\lesssim\|F\|_{\mathbb{W}}=\|f\|_{L^2(\Omega)},\quad \forall f\in L^2(\Omega).$$

By the same proof as Remark 2.1, we can deduce that the following regularity holds: for any $F=(0,f)$, $f\in L^2(\Omega)$, we have $(u^f,v^f)\in H^{1+r}(\Omega)\times H^{1+r}(\Omega)$ $(r>\frac{1}{2})$ and

$$\|u^f\|_{1+r}+\|v^f\|_{1+r}\le C\|f\|_0. \tag{3.9}$$

The discrete form for the problem (3.2) is to find $\vartheta_h=\beta-\lambda_h\in\mathbb{R}$, $U_h=(u_h,v_h)\in\mathbb{V}_h$, $\|kv_h\|_0=1$, such that

$$\mathcal{A}_h(U_h,\Psi_h)=\vartheta_h\mathcal{B}(U_h,\Psi_h),\quad \forall\Psi_h\in\mathbb{V}_h, \tag{3.10}$$

where

$$\begin{aligned}
\mathcal{A}_h(U_h,\Psi_h)=&\sum_{\kappa\in\pi_h}\int_\kappa\nabla u_h\cdot\nabla\psi_h dx+a\sum_{\kappa\in\pi_h}\int_\kappa\nabla v_h\cdot\nabla\varphi_h dx\\
&-\sum_{\kappa\in\pi_h}\int_\kappa k^2nu_h\psi_h dx+\sum_{\kappa\in\pi_h}\int_\kappa k^2\beta v_h\varphi_h dx,
\end{aligned} \tag{3.11a}$$

$$\mathcal{B}(U_h,\Psi_h)=\int_\Omega k^2v_h\varphi_h dx. \tag{3.11b}$$

The source problem associated with (3.10) is as follows: find $U_h^F=(u_h^f,v_h^f)\in\mathbb{V}_h$, such that

$$\mathcal{A}_h(U_h^F,\Psi_h)=\mathcal{B}(F,\Psi_h),\quad \forall\Psi_h\in\mathbb{V}_h. \tag{3.12}$$

The finite element problem associated with (3.8) is as follows: find $k_h^2\in\mathbb{C}$ and $U_h=(u_h,v_h)\in\mathbb{V}_h$, such that

$$\mathcal{A}_h(U_h,\Psi_h)=0,\quad \forall\Psi_h\in\mathbb{V}_h. \tag{3.13}$$

Lemma 3.2. *For any $a>0$, there exists a sufficiently large constant $M>0$ and a constant $C_3>0$ such that*

$$\mathcal{A}_h(\Psi_h,\Psi_h)+M\|\Psi_h\|_0^2\geq C_3\|\Psi_h\|_h^2,\quad \forall\Psi_h\in\mathbb{V}_h. \tag{3.14}$$

For any $-1<a<0$, there exists a sufficiently large constant $M>0$ and a constant $C_4>0$ such that

$$\mathcal{A}_h(\Psi_h,\mathbb{T}\Psi_h)+M\|\Psi_h\|_0^2\geq C_4\|\Psi_h\|_h^2,\quad \forall\Psi_h\in\mathbb{V}_h. \tag{3.15}$$

Proof. From the definitions of $\mathcal{A}_h(\cdot,\cdot)$ and $\|\cdot\|_h$, we deduce

$$\begin{aligned}\mathcal{A}_h(\Psi_h,\Psi_h)+M\|\Psi_h\|_0^2&=\sum_{\kappa\in\pi_h}\int_\kappa(\nabla\psi_h\cdot\nabla\psi_h+a\nabla\varphi_h\cdot\nabla\varphi_h-k^2n\psi_h^2+k^2\beta\varphi_h^2+M(\psi_h^2+\varphi_h^2))dx\\&=\sum_{\kappa\in\pi_h}\int_\kappa((\nabla\psi_h)^2+a(\nabla\varphi_h)^2+(M-k^2n)\psi_h^2+(M+k^2\beta)\varphi_h^2)dx\\&\geq\min\{1,a,M-k^2\|n\|_{0,\infty},M+k^2\beta\}(\|\psi_h\|_1^2+\|\varphi_h\|_1^2)\\&\geq C_3\|\Psi_h\|_h^2,\end{aligned}$$

where $C_3\leq\{1,a,M-k^2\|n\|_{0,\infty},M+k^2\beta\}$. Then (3.14) holds.

From the definition of $\mathcal{A}_h(\cdot,\cdot)$, (2.6) and Young's inequality, we can get (3.15). $\square$

It is easy to know that the following continuity property holds:

$$|\mathcal{A}_h(U_h,\Psi_h)|\lesssim\|U_h\|_h\|\Psi_h\|_h,\quad \forall U_h,\Psi_h\in\mathbb{V}_h.$$

Based on Theorem 2.1, we know that the eigenvalues of (3.13) converge to the eigenvalues of (3.8), hence, k^2 is not the eigenvalue of (3.13), thus (3.12) has the unique solution.

Since (3.12) is uniquely solvable, for each $F=(0,f)\in\mathbb{W}$, from [26] we can obtain the well-posedness of (3.12) and define the corresponding solution operator $\mathbb{K}_h=(S_h,T_h):L^2(\Omega)\to\mathbb{V}_h$ as follows:

$$\begin{aligned}&\mathcal{A}_h(\mathbb{K}_hf,\Psi_h)=\mathcal{B}(F,\Psi_h),\quad\forall\Psi_h\in\mathbb{V}_h,\\&\mathbb{K}_hf=U_h^F,\\&T_h:L^2(\Omega)\to\mathbb{V}_h,\quad T_hf=u_h^f,\\&S_h:L^2(\Omega)\to\mathbb{V}_h,\quad S_hf=v_h^f.\end{aligned}$$

Then (3.10) can be written as the following equivalent operator forms:

$$\vartheta_h T_h v_h = v_h, \quad S_h(\vartheta_h v_h) = u_h. \tag{3.16}$$

Let U^F and U_h^F be the solutions of (3.7) and (3.12), respectively. Define the consistency error term of nonconforming finite element:

$$E_h(U^F,\Psi_h) = \mathcal{A}_h(U^F,\Psi_h) - \mathcal{B}(F,\Psi_h), \quad \forall \Psi_h \in \mathbb{V}+\mathbb{V}_h. \tag{3.17}$$

From Lemma 3.2, we obtain the following lemma.

Lemma 3.3. *For any $F\in\mathbb{W}$, let $U^F\in(H^{1+r}(\Omega))^d$ be the solution of* (3.7)*, then*

$$E_h(U^F,\Psi_h) \lesssim h^r \|U^F\|_{1+r}\|\Psi_h\|_h, \quad \forall \Psi_h \in \mathbb{V}+\mathbb{V}_h. \tag{3.18}$$

Proof. Noting that $F=(0,f)$ and applying the same proof method as Lemma 2.2, we can get (3.18). □

Setting a bilinear form

$$\widehat{\mathcal{B}}(U,\Psi) = \int_\Omega k^2 U\cdot\Psi dx,$$

and letting $F=(g,f)$, we introduce a conjugate auxiliary problem: find $\widehat{U}^F\in\mathbb{V}$, such that

$$\mathcal{A}(\Psi,\widehat{U}^F) = \widehat{\mathcal{B}}(\Psi,F), \quad \forall \Psi\in\mathbb{V}. \tag{3.19}$$

Let $\widehat{U}^F$ be the solution of (3.19), then from Remark 2.1 we can know that it is valid that

$$\|\widehat{U}^F\|_{1+r} \lesssim \|F\|_{\mathbb{W}}. \tag{3.20}$$

Define the consistency error term of nonconforming finite element:

$$\widehat{E}_h(\Psi_h,\widehat{U}^F) = \mathcal{A}_h(\Psi_h,\widehat{U}^F) - \widehat{\mathcal{B}}(\Psi_h,F), \quad \forall \Psi_h \in \mathbb{V}+\mathbb{V}_h. \tag{3.21}$$

We use the arguement similar to $E_h(U^F,\Psi)$ to derive the following inequality

$$\widehat{E}_h(\Psi_h,\widehat{U}^F) \lesssim h^r\|\widehat{U}^F\|_{1+r}\|\Psi_h\|_h, \quad \forall \Psi_h\in\mathbb{V}+\mathbb{V}_h. \tag{3.22}$$

Lemma 3.4. *Let U^F and U_h^F be the solutions of* (3.7) *and* (3.12)*, respectively, then*

$$\|U^F - U_h^F\|_{\mathbb{W}} \lesssim h^r \|U^F - U_h^F\|_h. \tag{3.23}$$

Proof. Let $\mathbf{I}_h^c\widehat{U}^F \in (H^1(\Omega)\times H^1(\Omega))\cap\mathbb{V}_h$ be the linear interpolation of $\widehat{U}^F$ on π_h. Then, according to (3.7) and (3.12), we have $\mathcal{A}_h(U^F-U_h^F, I_h^c\widehat{U}^F)=0$, thus, from the interpolation error estimate, (3.21), (3.22) and (3.20) we obtain

$$\begin{aligned}
\widehat{\mathcal{B}}(U^F-U_h^F,F) =& \widehat{\mathcal{B}}(U^F-U_h^F,F) - \mathcal{A}_h(U^F-U_h^F,\widehat{U}^F) + \mathcal{A}_h(U^F-U_h^F,\widehat{U}^F) \\
=& -\widehat{E}_h(U^F-U_h^F,\widehat{U}^F) + \mathcal{A}_h(U^F-U_h^F,\widehat{U}^F - \mathbf{I}_h^c\widehat{U}^F) \\
\lesssim& h^r\|U^F-U_h^F\|_h\|\widehat{U}^F\|_{1+r} \\
\lesssim& h^r\|F\|_0\|U^F-U_h^F\|_h.
\end{aligned}$$

By using Riesz representation theorem, we obtain (3.23). □

Thanks to the Strang lemma, we get the following lemma.

Lemma 3.5. *Let U^F and U_h^F be the solutions of* (3.7) *and* (3.12)*, respectively. Then*

$$\|U^F - U_h^F\|_h \backsimeq \inf_{\Phi_h \in \mathbb{V}_h} \|U^F - \Phi_h\|_h + \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, Q_h)}{\|Q_h\|_h}. \tag{3.24}$$

Proof. Using the triangle inequality, for any $\Phi_h \in \mathbb{V}_h$,

$$\|U^F - U_h^F\|_h \leq \|U^F - \Phi_h\|_h + \|U_h^F - \Phi_h\|_h. \tag{3.25}$$

In the case of $-1 < a < 0$, from (3.15), (3.17) and the continuity property of $\mathcal{A}_h(\cdot,\cdot)$, we deduce

$$\begin{aligned}
\|U_h^F - \Phi_h\|_h^2 \lesssim & \mathcal{A}_h(U_h^F - \Phi_h, \mathbb{T}(U_h^F - \Phi_h)) + M\|U_h^F - \Phi_h\|_{\mathbb{W}}^2 \\
= & \mathcal{A}_h(U^F - U_h^F, \mathbb{T}(\Phi_h - U_h^F)) + \mathcal{A}_h(U^F - \Phi_h, \mathbb{T}(U_h^F - \Phi_h)) + M\|U_h^F - \Phi_h\|_{\mathbb{W}}^2 \\
= & \mathcal{A}_h(U^F, \mathbb{T}(\Phi_h - U_h^F)) - \mathcal{A}_h(U_h^F, \mathbb{T}(\Phi_h - U_h^F)) + \mathcal{A}_h(U^F - \Phi_h, \mathbb{T}(U_h^F - \Phi_h)) \\
& + M\|U_h^F - \Phi_h\|_{\mathbb{W}}^2 \\
\lesssim & E_h(U^F, \mathbb{T}(\Phi_h - U_h^F)) + \|U_h^F - \Phi_h\|_h \|U^F - \Phi_h\|_h + \|U_h^F - \Phi_h\|_{\mathbb{W}}^2.
\end{aligned} \tag{3.26}$$

When $\|U_h^F - \Phi_h\|_h \neq 0$, dividing both sides of (3.26) by $\|U_h^F - \Phi_h\|_h$, we obtain

$$\begin{aligned}
\|U_h^F - \Phi_h\|_h \lesssim & \frac{E_h(U^F, \mathbb{T}(\Phi_h - U_h^F))}{\|U_h^F - \Phi_h\|_h} + \|U^F - \Phi_h\|_h + \|U_h^F - \Phi_h\|_{\mathbb{W}} \\
\lesssim & \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, \mathbb{T}Q_h)}{\|Q_h\|_h} + \|U^F - \Phi_h\|_h + \|U_h^F - U^F\|_{\mathbb{W}}.
\end{aligned}$$

From the triangular inequality and (3.23), we get

$$\begin{aligned}
\|U^F - U_h^F\|_h \lesssim & \|U^F - \Phi_h\|_h + \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, \mathbb{T}Q_h)}{\|Q_h\|_h} + \|U_h^F - U^F\|_{\mathbb{W}} \\
\lesssim & \|U^F - \Phi_h\|_h + \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, Q_h)}{\|Q_h\|_h}, \quad \forall \Phi_h \in \mathbb{V}_h.
\end{aligned} \tag{3.27}$$

Using the continuity property of $\mathcal{A}_h(\cdot,\cdot)$ we have

$$\mathcal{A}_h(U^F - U_h^F, Q_h) \lesssim \|U^F - U_h^F\|_h \|Q_h\|_h, \quad \forall Q_h \in \mathbb{V}_h,$$

then

$$\begin{aligned}
\|U^F - U_h^F\|_h \gtrsim & \sup_{Q_h \in \mathbb{V}_h} \frac{\mathcal{A}_h(U^F, Q_h) - \mathcal{A}_h(U_h^F, Q_h)}{\|Q_h\|_h} \\
\gtrsim & \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, Q_h)}{\|Q_h\|_h}.
\end{aligned} \tag{3.28}$$

Noting that

$$\|U^F - U_h^F\|_h \geq \inf_{\Phi_h \in \mathbb{V}_h} \|U^F - \Phi_h\|_h,$$

and combining the above two inequalities we obtain

$$\|U^F - U_h^F\|_h \gtrsim \inf_{\Phi_h \in \mathbb{V}_h} \|U^F - \Phi_h\|_h + \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, Q_h)}{\|Q_h\|_h}. \tag{3.29}$$

From (3.27) and (3.29), we derive (3.24).

In the case of $a > 0$, similar to the proof of the case $-1 < a < 0$, taking $\mathbb{T}$ as the identity matrix, from (3.14), (3.17) and the continuity property of $\mathcal{A}_h(\cdot,\cdot)$, we can obtain (3.28). $\square$

From Lemmas 3.3 and 3.4, we have the following theorem.

Theorem 3.1. *Let $U^F \in (H^{1+r}(\Omega))^d$ $(r > \frac{1}{2})$ and U_h^F be the solutions of* (3.7) *and* (3.12), *respectively, then there holds the following estimate:*

$$\|U^F - U_h^F\|_h \lesssim h^r \|f\|_0. \tag{3.30}$$

Proof. From Lemma 3.5, Lemma 3.3 and interpolate error estimate, we have

$$\begin{aligned}
\|U^F - U_h^F\|_h &\lesssim \inf_{\Phi_h \in \mathbb{V}_h} \|U^F - \Phi_h\|_h + \sup_{Q_h \in \mathbb{V}_h} \frac{E_h(U^F, Q_h)}{\|Q_h\|_h} \\
&\lesssim \inf_{\Phi_h \in \mathbb{V}_h} \|U^F - \Phi_h\|_h + h^r \|U^F\|_{1+r} \\
&\lesssim h^r \|f\|_0.
\end{aligned}$$

The proof is completed. $\square$

From (3.30) we obtain

$$\|\mathbb{K}_h F\|_h \lesssim \|F\|_{\mathbb{W}} = \|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega). \tag{3.31}$$

Lemma 3.6. *There holds*

$$\|T - T_h\|_0 \to 0 \quad \textit{as} \;\; h \to 0. \tag{3.32}$$

Proof. From

$$\|T - T_h\|_0 = \sup_{f \in L^2(\Omega)} \frac{\|(T - T_h) f\|_0}{\|f\|_0},$$

(3.23) and (3.9), we get (3.32). $\square$

Lemma 3.7. *Let* (ϑ,U) *and* (ϑ_h,U_h) *be the jth eigenpairs of* (3.2) *and* (3.10)*, respectively. Then there holds the following identity*

$$\begin{aligned}\vartheta-\vartheta_h=&\mathcal{A}_h(U-U_h,U-U_h)-\vartheta_h\mathcal{B}(U-U_h,U-U_h)\\&+2\mathcal{A}_h(U-\mathbf{I}_hU,U_h)-2\vartheta_h\mathcal{B}(U-\mathbf{I}_hU,U_h).\end{aligned}\tag{3.33}$$

Proof. Since $\mathcal{B}(U,U)=1$ and $\mathcal{B}(U_h,U_h)=1$, we know

$$\mathcal{A}(U,U)=\vartheta,\quad \mathcal{A}_h(U_h,U_h)=\vartheta_h.$$

Thus,

$$\begin{aligned}\vartheta-\vartheta_h&=\mathcal{A}(U,U)+\mathcal{A}_h(U_h,U_h)-2\mathcal{A}_h(U_h,U_h)\\&=\mathcal{A}(U,U)+\mathcal{A}_h(U_h,U_h)-2\mathcal{A}_h(U,U_h)+2\mathcal{A}_h(U-U_h,U_h)\\&=\mathcal{A}_h(U-U_h,U-U_h)+2\mathcal{A}_h(U-U_h,U_h).\end{aligned}\tag{3.34}$$

Since

$$\mathcal{B}(U-U_h,U+U_h)=\mathcal{B}(U,U)+\mathcal{B}(U,U_h)-\mathcal{B}(U_h,U)-\mathcal{B}(U_h,U_h)=0,$$

then

$$\begin{aligned}\mathcal{B}(I_hU-U_h,U_h)&=\mathcal{B}(I_hU-U,U_h)+\mathcal{B}\Big(U-U_h,U_h-\frac{1}{2}U+\frac{1}{2}U\Big)-\frac{1}{2}\mathcal{B}(U-U_h,U+U_h)\\&=\mathcal{B}(I_hU-U,U_h)-\frac{1}{2}\mathcal{B}(U-U_h,U-U_h).\end{aligned}$$

Therefore,

$$\vartheta_h\mathcal{B}(I_hU-U_h,U_h)=\vartheta_h\mathcal{B}(I_hU-U,U_h)-\frac{1}{2}\vartheta_h\mathcal{B}(U-U_h,U-U_h).$$

Together with (3.10), we deduce

$$\begin{aligned}\mathcal{A}_h(U-U_h,U_h)&=\mathcal{A}_h(U-I_hU,U_h)+\mathcal{A}_h(I_hU-U_h,U_h)\\&=\mathcal{A}_h(U-I_hU,U_h)+\vartheta_h\mathcal{B}(I_hU-U_h,U_h)\\&=\mathcal{A}_h(U-I_hU,U_h)+\vartheta_h\mathcal{B}(I_hU-U,U_h)-\frac{1}{2}\vartheta_h\mathcal{B}(U-U_h,U-U_h).\end{aligned}\tag{3.35}$$

Substituting (3.35) into (3.34), we get (3.33). □

Theorem 3.2. *Let* $(\vartheta,U=(u,v))$ *and* $(\vartheta_h,U_h=(u_h,v_h))$ *be the eigenpairs of* (3.2) *and* (3.10)*, respectively, then it is valid the following estimates:*

$$\|U-U_h\|_h\lesssim h^r,\tag{3.36a}$$

$$\|U-U_h\|_{\mathbb{W}}\lesssim h^r\|U-U_h\|_h,\tag{3.36b}$$

$$|\vartheta-\vartheta_h|\lesssim h^{2r}.\tag{3.36c}$$

Proof. Since $\|T-T_h\|_0\to 0$ $(h\to 0)$, from the spectral approximation theory [32] and (3.30), we have

$$\|v-v_h\|_0+|\vartheta-\vartheta_h|\lesssim\|(T_h-T)v\|_0\lesssim h^r. \tag{3.37}$$

From (3.31), Theorem 3.1 and (3.37), we deduce

$$\begin{aligned}\|U-U_h\|_h&=\|\vartheta\mathbb{K}v-\vartheta_h\mathbb{K}_hv_h\|_h\\&\le\|\vartheta\mathbb{K}v-\vartheta\mathbb{K}_hv\|_h+\|\vartheta\mathbb{K}_hv-\vartheta_h\mathbb{K}_hv_h\|_h\\&\lesssim\|\mathbb{K}v-\mathbb{K}_hv\|_h+|\vartheta-\vartheta_h|+\|v-v_h\|_0\lesssim h^r.\end{aligned}$$

Then (3.36a) holds. From the triangle inequality, (3.31) and (3.37), we get

$$\begin{aligned}\big|\|U-U_h\|_h-\|\vartheta\mathbb{K}v-\vartheta\mathbb{K}_hv\|_h\big|&\le\|\vartheta\mathbb{K}v-\vartheta_h\mathbb{K}_hv_h-\vartheta\mathbb{K}v+\vartheta\mathbb{K}_hv\|_h\\&=\|\mathbb{K}_h(\vartheta v-\vartheta_hv_h)\|_h\\&\lesssim\|\vartheta v-\vartheta_hv_h\|_0\\&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)v\|_h,\end{aligned} \tag{3.38}$$

that is to say

$$\|U-U_h\|_h\backsimeq\|\vartheta\mathbb{K}v-\vartheta\mathbb{K}_hv\|_h. \tag{3.39}$$

From (3.16), (3.23) and (3.37), we can derive

$$\begin{aligned}\|U-U_h\|_{\mathbb{W}}&=\|\mathbb{K}\vartheta v-\mathbb{K}_h\vartheta_hv_h\|_{\mathbb{W}}\\&\le\|(\mathbb{K}-\mathbb{K}_h)\vartheta v\|_{\mathbb{W}}+\|\mathbb{K}_h\vartheta v-\mathbb{K}_h\vartheta_hv_h\|_{\mathbb{W}}\\&\lesssim\|(\mathbb{K}-\mathbb{K}_h)\vartheta v\|_{\mathbb{W}}+\|\vartheta v-\vartheta_hv_h\|_0\\&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)v\|_h+\|(T_h-T)v\|_0\\&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)v\|_h+h^r\|(\mathbb{K}-\mathbb{K}_h)U\|_h\\&\lesssim h^r\|(\mathbb{K}-\mathbb{K}_h)v\|_h.\end{aligned} \tag{3.40}$$

Combining (3.39) with (3.40), we obtain (3.36b).

Substituting (3.36a), (3.36b), (2.18b) and (2.19) into (3.33), together with

$$2\vartheta_h\mathcal{B}(U-\mathbf{I}_hU,U_h)\le Ch^{1+r},$$

we obtain (3.36c). □

Theorem 3.3. *Let $a>0$, (ϑ,U) and (ϑ_h,U_h) be the jth eigenpairs of* (3.2) *and* (3.10), *respectively. $\lambda=\beta-\vartheta$ and $\lambda_h=\beta-\vartheta_h$. Assume that $u,v\in H^{1+r}(\Omega)$ and h is sufficiently small. When the eigenfunctions are singular and $\|U-U_h\|_h\ge Ch^{1-\delta}$ $(\delta>0)$ for the CR element and $\|U-U_h\|_h\ge Ch^{1+r-\frac{1}{2}}$ for the ECR element, it is valid that*

$$\lambda_h\ge\lambda. \tag{3.41}$$

Proof. From (3.33), $\lambda=\beta-\vartheta$ and $\lambda_h=\beta-\vartheta_h$, (3.11a), (2.19) and (3.11b) we deduce

$$\begin{aligned}\lambda_h-\lambda=&\beta-\lambda-(\beta-\lambda_h)=\vartheta-\vartheta_h\\ =&\mathcal{A}_h(U-U_h,U-U_h)-\vartheta_h\mathcal{B}(U-U_h,U-U_h)+2\mathcal{A}_h(U-\mathbf{I}_hU,U_h)\\ &-2\vartheta_h\mathcal{B}(U-\mathbf{I}_hU,U_h)\\ =&\sum_{\kappa\in\pi_h}\int_\kappa\nabla(u-u_h)\cdot\nabla(u-u_h)dx+a\sum_{\kappa\in\pi_h}\int_\kappa\nabla(v-v_h)\cdot\nabla(v-v_h)dx\\ &-\sum_{\kappa\in\pi_h}\int_\kappa k^2n(u-u_h)^2dx+\sum_{\kappa\in\pi_h}\int_\kappa k^2\beta(v-v_h)^2dx-\vartheta_h\mathcal{B}(U-U_h,U-U_h)\\ &-2\sum_{\kappa\in\pi_h}k^2n\int_\kappa(u-I_hu)u_hdx+2\sum_{\kappa\in\pi_h}k^2\beta\int_\kappa(v-I_hv)v_hdx\\ &-2\vartheta_hk^2\int_\Omega(v-I_hv)v_hdx.\end{aligned}\tag{3.42}$$

We analyse each term on the right hand side of (3.42). For the third, the fourth and the fifth terms on the right hand side of (3.42), from (3.36b), we know that $\|U-U_h\|^2_{\mathbb{W}}$ is a small quantity of higher order compared with $\|U-U_h\|_h^2$. For the sixth, the seventh and the eighth terms on the right hand side of (3.42), we take $\int_\kappa(u-I_hu)u_hdx$ as an example for analysis.

For the CR element, there holds

$$\int_\kappa(u-I_hu)u_hdx\lesssim h|u-I_hu|_{1,\kappa}\lesssim h|u-u_h|_{1,\kappa},\tag{3.43}$$

and for the ECR element,

$$\int_\kappa(u-I_hu)u_hdx=\int_\kappa(u-I_hu)(u_h-P_0^\kappa u_h)dx\lesssim h_\kappa^{2+r}\|u\|_{1+r,\kappa}\|u_h\|_{1,\kappa}.\tag{3.44}$$

Combining (3.43) with (3.44), we obtain that the sixth, the seventh and the eighth terms on the right hand side of (3.42) are small quantities of higher order compared with $\|U-U_h\|_h^2$.

According to the above analysis, the first and the second terms on the right hand side of (3.42) are dominant terms and positive when $a>0$. Thus (3.41) holds. □

For the mTEP, from Theorem 3.3 we know if $a>0$, the CR element and the ECR element method approximate the exact eigenvalues from above.

Remark 3.1. When $\beta-\lambda_h^c=\vartheta_h^c$ is an approximate eigenvalue of (3.2) by conforming finite element method, thanks to the Lemma 9.1 in [32], we get

$$\begin{aligned}\lambda-\lambda_h^c=&(\beta-\lambda_h^c)-(\beta-\lambda)=\vartheta_h^c-\vartheta\\ =&\mathcal{A}_h(U-U_h,U-U_h)-\vartheta\mathcal{B}(U-U_h,U-U_h).\end{aligned}\tag{3.45}$$

From (3.45), we can deduce that the eigenvalue λ_h^c by the conforming finite element method approximate the exact eigenvalue from below for the mTEP (3.1) with $a>0$.

Remark 3.2. In Theorems 2.2 and 3.2, the constants implied in the inequalities are closely related to the shape of grid and the distribution of eigenvalues. Further quantitative analysis for these constants in error estimation will be an interesting issue that worth studying.

4 Numerical experiments

Our program is compiled under the package of iFEM [36] and we use the internal command 'eigs' in MATLAB 2018a to solve generalized matrix eigenvalue problem on a ThinkPad laptop with 32G memory.

In our computation, the test domains are set to be the unit square domain $\Omega_S=(0,1)^2$ and the L-shaped domain $\Omega_L=(-1,1)^2\backslash[0,1)\times(-1,0]$ for two-dimensional case, and the Fichera domain $\Omega_F=(-1,1)^3\backslash[0,1]^3$ and the unit cubic domain $\Omega_C=(0,1)^3$ for three-dimensional case. We adopt a uniform mesh π_h for each domain.

4.1 The numerical results of the TEP (2.2)

The coefficient matrix $A(x)$ and the index of refraction $n(x)$ of the problem (2.2) are chosen as follows:

$$\begin{aligned}
&\text{Case 1}: \quad A(x)=\begin{pmatrix} 2+x_1^2 & x_1x_2 \\ & \\ x_1x_2 & 2+x_2^2 \end{pmatrix}, && n(x)=2+|x_1+x_2|,\\
&\text{Case 2}: \quad A(x)=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, && n(x)=3.
\end{aligned}$$

We denote the approximate eigenvalue as $k_{j,h}=\sqrt{\lambda_{j,h}-1}$.

Since the exact eigenvalues of the problem on all test domains are unknown, in order to investigate the convergence behavior, we use the following formula

$$order(k_{j,h})\approx\frac{1}{(\lg 2)}\lg\left|\frac{k_{j,h}-k_{j,\frac{h}{2}}}{k_{j,\frac{h}{2}}-k_{j,\frac{h}{4}}}\right|$$

to compute the approximate convergence order.

Theorem 2.2 shows that when $\alpha=1$, the theoretical convergence order of the approximate eigenvalues is $2r$ under the assumption of the eigenfunction's smoothness index $r\in(\frac{1}{2},1]$. From Tables 1-2, we see that the convergence order of the approximate eigenvalues is 2 in the unit square, and from Tables 3-4 we see that the convergence order of the first approximate eigenvalue is about 1.33 and the convergence order of the sixth and the seventh approximate eigenvalues is about 1.4 in the L-shaped domain. These numerical results show that the regularity assumption of Theorem 2.2 is reasonable and are consistent with the theoretical analysis.

Table 1: The approximate eigenvalues of (2.2) by the CR element in the unit square for Case 1.

h	$k_{1,h}$	order	$k_{j,h}$ $(j=5,6)$	order
$\sqrt{2}/16$	2.309789		$4.932109 \mp 0.246840i$	
$\sqrt{2}/32$	2.310801		$4.938256 \mp 0.243196i$	
$\sqrt{2}/64$	2.311054	2.01	$4.939777 \mp 0.242264i$	2.00
$\sqrt{2}/128$	2.311117	2.00	$4.940156 \mp 0.242030i$	2.00
$\sqrt{2}/256$	2.311132	2.00	$4.940250 \mp 0.241971i$	2.00
$\sqrt{2}/512$	2.311136	2.00	$4.940274 \mp 0.241956i$	2.00

Table 2: The approximate eigenvalues of (2.2) by the ECR element in the unit square for Case 1.

h	$k_{1,h}$	order	$k_{j,h}$ $(j=5,6)$	order
$\sqrt{2}/16$	2.308692		$4.924386 \mp 0.248306i$	
$\sqrt{2}/32$	2.310527		$4.936303 \mp 0.243599i$	
$\sqrt{2}/64$	2.310985	2.00	$4.939287 \mp 0.242367i$	1.99
$\sqrt{2}/128$	2.311099	2.00	$4.940033 \mp 0.242056i$	2.00
$\sqrt{2}/256$	2.311128	2.00	$4.940220 \mp 0.241978i$	2.00
$\sqrt{2}/512$	2.311135	2.00	$4.940266 \mp 0.241958i$	2.00

Table 3: The approximate eigenvalues of (2.2) by the CR element in the L-shaped domain for Case 1.

h	$k_{1,h}$	order	$k_{j,h}$ $(j=6,7)$	order
$\sqrt{2}/16$	0.871160		$3.037828 \mp 0.083515i$	
$\sqrt{2}/32$	0.872855		$3.042374 \mp 0.082935i$	
$\sqrt{2}/64$	0.873528	1.33	$3.043940 \mp 0.082639i$	1.52
$\sqrt{2}/128$	0.873795	1.33	$3.044502 \mp 0.082507i$	1.46
$\sqrt{2}/256$	0.873901	1.33	$3.044710 \mp 0.082451i$	1.42
$\sqrt{2}/512$	0.873943	1.33	$3.044789 \mp 0.082428i$	1.39

Table 4: The approximate eigenvalues of (2.2) by the ECR element in the L-shaped domain for Case 1.

h	$k_{1,h}$	order	$k_{j,h}(j=6,7)$	order
$\sqrt{2}/16$	0.871038		$3.036014 \mp 0.083225i$	
$\sqrt{2}/32$	0.872825		$3.041918 \mp 0.082859i$	
$\sqrt{2}/64$	0.873521	1.36	$3.043826 \mp 0.082620i$	1.62
$\sqrt{2}/128$	0.873793	1.35	$3.044473 \mp 0.082502i$	1.55
$\sqrt{2}/256$	0.873901	1.35	$3.044703 \mp 0.082450i$	1.48
$\sqrt{2}/512$	0.873943	1.34	$3.044788 \mp 0.082428i$	1.43

The numerical results in the three-dimensional domains are listed in Tables 5-6 We observe from these tables that the numerical results also coincide with the theoretical analysis.

Table 5: The approximate eigenvalues of (2.2) in the unit cubic for Case 2.

	CR				ECR			
h	$k_{1,h}$	order	$k_{4,h}$	order	$k_{1,h}$	order	$k_{4,h}$	order
$\sqrt{3}/2$	2.90609		4.16623		2.88316		4.10695	
$\sqrt{3}/4$	3.08309		4.35316		3.07592		4.33351	
$\sqrt{3}/8$	3.12700	2.01	4.42063	1.47	3.12509	1.97	4.41528	1.47
$\sqrt{3}/16$	3.13795	2.00	4.43733	2.01	3.13746	1.99	4.43596	1.98

Table 6: The approximate eigenvalues of (2.2) in the Fichera domain for Case 2.

	CR				ECR			
h	$k_{1,h}$	order	$k_{4,h}$	order	$k_{1,h}$	order	$k_{4,h}$	order
$\sqrt{3}/2$	1.34055		2.06230		1.33818		2.05389	
$\sqrt{3}/4$	1.37441		2.09922		1.37376		2.09693	
$\sqrt{3}/8$	1.38577	1.57	2.10982	1.80	1.38561	1.59	2.10924	1.81

4.2 The numerical results of the mTEP (3.1)

The numerical results in the two-dimensional domains are listed in Tables 7-12 and the results in the three-dimensional domains are listed in Tables 13-16. From these tables we observe that the eigenvalues obtained by the CR element and the ECR element decreases gradually, the eigenvalues obtained by the conforming P_1 element (P_1) increase gradually as the mesh size h decreases, and the eigenvalues obtained by the conforming and nonconforming elements get closer and closer as h decreases. These numerical results tell us that for the mTEP, the eigenvalues obtained by the CR element and the ECR element can provide the upper bound and the conforming element eigenvalues give the lower bound for the exact eigenvalues. So we obtain an interval where the exact eigenvalues are located.

Table 7: The approximate eigenvalues of (3.1) in the unit square when $n=4$, $a=1$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.143060	-138.7974	4.143516	-138.5422	4.140818	-140.0526
$\sqrt{2}/32$	4.142773	-139.1044	4.142887	-139.0401	4.142206	-139.4203
$\sqrt{2}/64$	4.142698	-139.1820	4.142727	-139.1659	4.142556	-139.2612
$\sqrt{2}/128$	4.142680	-139.2015	4.142687	-139.1975	4.142644	-139.2213
$\sqrt{2}/256$	4.142675	-139.2064	4.142677	-139.2054	4.142666	-139.2113
$\sqrt{2}/512$	4.142674	-139.2076	4.142674	-139.2073	4.142671	-139.2088
$\sqrt{2}/1024$	4.142673	-139.2079	4.142673	-139.2078	4.142673	-139.2082
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 8: The approximate eigenvalues of (3.1) in the L-shaped domain when $n=4$, $a=1$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.302978	-19.2676	4.303470	-19.2625	4.297340	-19.5556
$\sqrt{2}/32$	4.301555	-19.3531	4.301678	-19.3518	4.299670	-19.4628
$\sqrt{2}/64$	4.301055	-19.3859	4.301086	-19.3856	4.300397	-19.4283
$\sqrt{2}/128$	4.300874	-19.3986	4.300881	-19.3985	4.300635	-19.4151
$\sqrt{2}/256$	4.300806	-19.4036	4.300808	-19.4035	4.300717	-19.4101
$\sqrt{2}/512$	4.300780	-19.4055	4.300781	-19.4055	4.300746	-19.4081
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 9: The approximate eigenvalues of (3.1) in the unit square when $n=4$, $a=2$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.1067042	-223.1881	4.1070405	-222.8579	4.1050515	-224.8255
$\sqrt{2}/32$	4.1064942	-223.6210	4.1065783	-223.5379	4.1060760	-224.0327
$\sqrt{2}/64$	4.1064396	-223.7301	4.1064606	-223.7093	4.1063347	-223.8332
$\sqrt{2}/128$	4.1064258	-223.7575	4.1064310	-223.7523	4.1063995	-223.7833
$\sqrt{2}/256$	4.1064223	-223.7643	4.1064236	-223.7630	4.1064157	-223.7708
$\sqrt{2}/512$	4.1064214	-223.7660	4.1064217	-223.7657	4.1064198	-223.7676
$\sqrt{2}/1024$	4.1064212	-223.7664	4.1064213	-223.7664	4.1064208	-223.7668
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 10: The approximate eigenvalues of (3.1) in the L-shaped domain when $n=4$, $a=2$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.224531	-31.4009	4.224887	-31.3942	4.220420	-31.8449
$\sqrt{2}/32$	4.223492	-31.5337	4.223581	-31.5320	4.222116	-31.7037
$\sqrt{2}/64$	4.223127	-31.5848	4.223149	-31.5843	4.222646	-31.6506
$\sqrt{2}/128$	4.222994	-31.6046	4.223000	-31.6045	4.222819	-31.6304
$\sqrt{2}/256$	4.222945	-31.6124	4.222946	-31.6123	4.222879	-31.6225
$\sqrt{2}/512$	4.222926	-31.6154	4.222926	-31.6154	4.222901	-31.6194
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

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Table 11: The approximate eigenvalues of (3.1) in the unit square when $n=4$, $a=4$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.0886746	-386.1024	4.0889524	-385.6081	4.0873091	-388.5660
$\sqrt{2}/32$	4.0885016	-386.7908	4.0885710	-386.6665	4.0881561	-387.4094
$\sqrt{2}/64$	4.0884566	-386.9637	4.0884740	-386.9326	4.0883700	-387.1186
$\sqrt{2}/128$	4.0884453	-387.0070	4.0884496	-386.9992	4.0884236	-387.0458
$\sqrt{2}/256$	4.0884424	-387.0178	4.0884435	-387.0159	4.0884370	-387.0275
$\sqrt{2}/512$	4.0884417	-387.0206	4.0884419	-387.0201	4.0884403	-387.0230
$\sqrt{2}/1024$	4.0884416	-387.0212	4.0884414	-387.0211	4.0884412	-387.0218
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 12: The approximate eigenvalues of (3.1) in the L-shaped domain when $n=4$, $a=4$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{2}/16$	4.18599	-55.42586	4.18628	-55.41547	4.18261	-56.17579
$\sqrt{2}/32$	4.18514	-55.65105	4.18521	-55.64843	4.18400	-55.93885
$\sqrt{2}/64$	4.18484	-55.73770	4.18485	-55.73705	4.18444	-55.84948
$\sqrt{2}/128$	4.18473	-55.77139	4.18473	-55.77123	4.18458	-55.81514
$\sqrt{2}/256$	4.18469	-55.78458	4.18469	-55.78454	4.18463	-55.80180
$\sqrt{2}/512$	4.18467	-55.78977	4.18467	-55.78976	4.18465	-55.79657
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 13: The approximate eigenvalues of (3.1) in the unit cubic when $n=4$, $a=2$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{3}/2$	4.1037	-126.7935	4.0792	-186.8073	4.0154	-276.7109
$\sqrt{3}/4$	4.0755	-188.6903	4.0663	-216.1854	4.0418	-246.4540
$\sqrt{3}/8$	4.0662	-214.0709	4.0622	-224.4324	4.0543	-233.6343
$\sqrt{3}/16$	4.0624	-222.7847	4.0608	-226.6239	4.0587	-229.2015
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 14: The approximate eigenvalues of (3.1) in the unit cubic when $n=4$, $a=4$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{3}/2$	4.0762	-221.5238	4.0659	-324.6261	4.0129	-463.8516
$\sqrt{3}/4$	4.0596	-327.9567	4.0552	-372.4433	4.0348	-418.3474
$\sqrt{3}/8$	4.0537	-370.1977	4.0517	-385.5676	4.0452	-399.4757
$\sqrt{3}/16$	4.0514	-383.5621	4.0506	-388.9896	4.0488	-392.8896
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 15: The approximate eigenvalues of (3.1) in the Fichera domain when $n=4$, $a=2$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{3}/2$	4.2148	-38.5242	4.2114	-38.9662	4.0841	-51.2581
$\sqrt{3}/4$	4.1985	-41.4659	4.1883	-41.3289	4.1337	-46.3939
$\sqrt{3}/8$	4.1846	-42.1087	4.1799	-42.1342	4.1594	-44.0269
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

Table 16: The approximate eigenvalues of (3.1) in the Fichera domain when $n=4$, $a=4$, $k=0.35$.

	CR		ECR		P_1	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
$\sqrt{3}/2$	4.1752	-67.6999	4.1753	-68.3453	4.0701	-88.4833
$\sqrt{3}/4$	4.1609	-72.7274	4.1563	-72.2839	4.1111	-80.6471
$\sqrt{3}/8$	4.1516	-73.6870	4.1493	-73.6227	4.1324	-76.7655
trend	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

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