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Fully Discrete Schemes with First- and Second-Order Temporal Accuracy for the Incompressible Magnetohydrodynamic Flow Based on the Generalized Scalar Auxiliary Variable Approach

Huimin Ma and Pengzhan Huang*

College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830017, China

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Abstract. Based on the generalized scalar auxiliary variable approach and vector penalty projection method, some fully discrete schemes with first- and second-order accuracy in time direction are constructed for solving the incompressible magnetohydrodynamic model. It is a combination of mixed finite element approximation for spatial discretization and first-order backward Euler/second-order backward differential formula for temporal discretization. The proposed schemes own several features: it decouples unknown physical variables and linearizes the nonlinear terms, then it only needs to solve some linear equations at each temporal level; although the divergence of numerical velocity is not exactly equal to zero, it can approximately meet the mass conservation when one takes small penalty parameter; while the computation of the velocity and pressure are decoupled, numerical results show that the velocity and pressure can reach second-order accuracy in time. The resulting schemes are supported by numerical analysis and simulation.

AMS subject classifications: 65M60, 65M12

Key words: Magnetohydrodynamic model, stability analysis, generalized scalar auxiliary variable, vector penalty projection.

1 Introduction

The incompressible magnetohydrodynamic (MHD) model, which is comprised of the incompressible Navier-Stokes equations and the Maxwell equations via Lorentz force and

*Corresponding author.
Emails: huiminmhm@163.com (H. Ma), hpzh@xju.edu.cn (P. Huang)

Ohm's law, is commonly used to describe the interaction between a viscous, incompressible, electrically conducting fluid and an external magnetic field. It has important applications in fusion technology, submarine propulsion system, liquid metals in magnetic pumps and so on [13,18,42].

In this paper, we consider the following time-dependent MHD equations. Given a bounded and regular domain $\Omega\subset\mathbb{R}^d$, $d=2$ or 3, and for a final time $T>0$, find the velocity field $\mathbf{u}:(0,T]\times\Omega\to\mathbb{R}^d$, the pressure $p:(0,T]\times\Omega\to\mathbb{R}$ and the magnetic field $\mathbf{H}:(0,T]\times\Omega\to\mathbb{R}^d$ satisfying [21,42]

$$
\begin{aligned}
&\mathbf{u}_t-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f}-(\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\mathbf{H}\times\mathrm{curl}\mathbf{H} && \text{in } \Omega\times(0,T], && (1.1a)\\
&\mathrm{div}\mathbf{u}=0 && \text{in } \Omega\times(0,T], && (1.1b)\\
&\mu\mathbf{H}_t+\sigma^{-1}\mathrm{curlcurl}\mathbf{H}=\sigma^{-1}\mathrm{curl}\mathbf{g}+\mu\mathrm{curl}(\mathbf{u}\times\mathbf{H}) && \text{in } \Omega\times(0,T], && (1.1c)\\
&\mathrm{div}\mathbf{H}=0 && \text{in } \Omega\times(0,T], && (1.1d)
\end{aligned}
$$

with initial data and homogeneous boundary condition

$$
\begin{aligned}
&\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\quad \mathbf{H}(\mathbf{x},0)=\mathbf{H}_0(\mathbf{x})\quad \text{with } \mathrm{div}\mathbf{u}_0=0,\quad \mathrm{div}\mathbf{H}_0=0\quad \text{in } \Omega, && (1.2a)\\
&\mathbf{u}|_{S_T}=0,\quad \mathbf{H}\cdot\mathbf{n}|_{S_T}=0,\quad \mathbf{n}\times\mathrm{curl}\mathbf{H}|_{S_T}=0, && (1.2b)
\end{aligned}
$$

where $S_T:=\partial\Omega\times[0,T]$ and $\mathbf{n}$ represents the unit outward normal of the boundary $\partial\Omega$. The model has three physical parameters: ν is the kinematic viscosity, μ is the magnetic permeability and σ is the electric conductivity. The vector-value functions $\mathbf{g}$ and $\mathbf{f}$ represent the known applied current satisfying $(\mathbf{n}\times\mathbf{g})|_{S_T}=0$ and the external force, respectively.

Numerical approximation of the MHD model is challenging, because it is a system with nonlinear terms, coupling of multi-physics fields and divergence-free. Faced with those challenges, researchers have designed a series of efficient numerical schemes by applying the finite element method [17, 64], finite difference method [8, 12], finite volume method [10,45], Fourier spectral method [19] for spatial discretization and the first-order Euler semi-implicit scheme [17, 22, 63, 65], the first-order Euler implicit/explicit scheme [59], Crank-Nicolson scheme [15, 24, 39, 51, 69], second-order backward differential formula [34, 60, 74], projection type scheme [20, 46, 52, 62, 71, 72], time filter scheme [11,25], blended backward differential formula scheme [36,37], deferred correction scheme [16] and so on. One of the challenges in numerically solving the MHD equations is the treatment of nonlinear terms, which can be divided into several types: fully implicit, semi-implicit and explicit schemes. It is known that when the convection and Lorentz force terms are treated by fully implicit and semi-implicit schemes, the velocity and magnetic field are not completely decoupled, which will increase the computational complexity. Compared to the above schemes, the explicit scheme [38,44,59] can decouple the velocity and magnetic field at each time step. Then the computational scale and the storage times of matrix are reduced. Usually, it is conditionally stable. Furthermore, Zhang et al. [66,67] proposed first-order fully decoupled scheme, which is unconditionally energy stable.