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Energy Conservative Local Discontinuous Galerkin Methods for the Euler-Korteweg Equations

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Abstract. In this paper, we develop through a careful selection of the auxiliary variables and numerical fluxes an energy conservative local discontinuous Galerkin (LDG) method based on a hybrid form of the general Euler-Korteweg (EK) equations with a variable capillarity coefficient. This energy conservative LDG discretization is of optimal order of accuracy for alternating numerical fluxes, but not for central numerical fluxes which result in the reduction of one order of accuracy when odd degree polynomial basis functions are used. Also, a relatively simple energy conservative LDG discretization for the EK-equations with an irrotational velocity field is presented. Due to the presence of a highly nonlinear third-order spatial derivative term, which originates from the divergence of the Korteweg stress tensor, we employ the novel semi-implicit spectral deferred correction (SDC) method as temporal discretization. The SDC method can be applied to highly nonlinear ordinary differential equations (ODEs) without separating stiff and non-stiff components and is numerically stable for a time step proportional to the mesh size. Numerical experiments, including ones with adaptive meshes, are performed to illustrate the accuracy and capability of the proposed methods to solve the EK-equations.

AMS subject classifications: 65M60, 35Q35

Key words: Euler-Korteweg equations, energy conservation, local discontinuous Galerkin methods, spectral deferred correction methods.

1 Introduction

In this work, we concentrate on the design of local discontinuous Galerkin (LDG) methods for the Euler-Korteweg (EK) equations with a variable capillarity coefficient, coupled

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with a novel semi-implicit spectral deferred correction (SDC) time discretization method. These equations take the surface tension of interfaces into consideration using a capillarity coefficient when modeling the dynamics of an isothermal compressible fluid, and read [5]

$$\rho_t+\nabla\cdot(\rho\mathbf{u})=0, \tag{1.1a}$$
$$(\rho\mathbf{u})_t+\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}+p\underline{I})=\nabla\cdot\underline{K}, \tag{1.1b}$$

in $\Omega\times(0,T]$, $\Omega\subset\mathbb{R}^d$, $d\leq 3$, $T>0$, where ρ is the density, $\mathbf{u}\in\mathbb{R}^d$ the fluid velocity, $p=p(\rho)$ the pressure, $\underline{I}$ the identity matrix and $\otimes$ the tensor product. The matrix $\underline{K}$ denotes the Korteweg stress tensor and is defined as

$$\underline{K}=\left(\rho\kappa(\rho)\Delta\rho+\frac{1}{2}(\rho\kappa'(\rho)+\kappa(\rho))|\nabla\rho|^2\right)\underline{I}-\kappa(\rho)\nabla\rho\otimes\nabla\rho, \tag{1.2}$$

with $\kappa=\kappa(\rho)$ the capillarity coefficient. The highly nonlinear third order dispersive term $\nabla\cdot\underline{K}$ can be reduced to the simplified form

$$\nabla\cdot\underline{K}=\rho\nabla\left(\kappa(\rho)\Delta\rho+\frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\right).$$

The EK-equations (1.1) then can be written in an equivalent non-conservative form as

$$\rho_t+\nabla\cdot(\rho\mathbf{u})=0, \tag{1.3a}$$
$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla F'(\rho)=\nabla\left(\kappa(\rho)\Delta\rho+\frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\right), \tag{1.3b}$$

with $F=F(\rho)$ the free energy density, which relates to the pressure p through the relation

$$p(\rho)=\rho F'(\rho)-F(\rho).$$

For a capillary fluid, unlike a regular isothermal fluid, the total energy on Ω takes the form

$$\mathcal{H}=\int_\Omega\left(\frac{1}{2}\rho|\mathbf{u}|^2+F(\rho)+\frac{1}{2}\kappa(\rho)|\nabla\rho|^2\right)d\mathbf{x}, \tag{1.4}$$

which shows that regions with large density gradients can contribute significantly to the total energy.

The EK-equations can be endowed with a non-monotone van der Waals equation of state [35]

$$p(\rho)=Rb\frac{\theta\rho}{b-\rho}-a\rho^2, \tag{1.5}$$

with temperature θ, universal gas constant R and positive constants a, b depending on the fluid, which allows the modelling of liquid-vapor mixtures undergoing a phase transition, i.e., there is mass transfer driven by thermodynamics between the phases. As a