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A Fast Cascadic Multigrid Method for Direct Finite Difference Discretizations of 3D Biharmonic Equations on Rectangular Domains

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Abstract. A new extrapolation cascadic multigrid (EXCMG) method is developed to solve large sparse symmetric positive definite systems resulting from the classical 25-point finite-difference discretizations of the three-dimensional (3D) biharmonic equation on rectangular domains. We accomplish this by designing a quartic interpolation-based prolongation operator and using the symmetric successive over-relaxation (SSOR) preconditioned CG method as the multigrid smoother. For the new prolongation operator, quartic interpolations are used for the finite difference solutions on coarse and fine grids twice and once so that two approximations can be obtained on the next finer grid, and then the completed Richardson extrapolation is used for these two approximations to obtain an excellent initial guess on the next finer grid. The proposed EXCMG method with the new prolongation operator is easier to implement than the original EXCMG method. Numerical experiments demonstrate that the new EXCMG is a highly efficient solver for the 3D biharmonic equation and is considerably faster than the original EXCMG method and the aggregation-based algebraic multigrid (AGMG) method developed by Y. Notay. The proposed EXCMG method can solve discrete 3D biharmonic equations with more than 100 million unknowns in dozens of seconds.

AMS subject classifications: 65M10, 78A48

Key words: Biharmonic equation, cascadic multigrid, quartic interpolation, high efficiency, Richardson extrapolation.

1 Introduction

A class of fourth-order partial differential equations, known as the biharmonic equation, has emerged from fields such as continuum mechanics, fluid mechanics, geohydrodynamics, and geophysics. These equations encompass linear elasticity theory, phase-field

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simulations, and Navier-Stokes flows. Due to their significant practical applications, numerous numerical methods have been proposed for solving these equations [1–15]. However, most of these works focus on the two-dimensional cases, with limited attention given to 3D biharmonic equations. This is due to the fact that the ill-conditioning of the discrete biharmonic problems is more severe than that of the discrete problems in the second-order systems. The condition number increases at a rate of $\mathcal{O}(h^{-4})$ rather than $\mathcal{O}(h^{-2})$ for the second-order problems (where h is the meshsize in the discretization). Thus, solving 3D biharmonic problems requires significant computational power and memory storage [8,12]. This work aims to develop a fast solver that can handle the ill-conditioning of discrete 3D biharmonic problems.

In this paper, we will consider the numerical solution for the following 3D biharmonic equation in a unit cube:

$$\Delta^2 p(x,y,z) = f(x,y,z), \quad (x,y,z) \in \Omega = [0,1]^3, \tag{1.1}$$

with Dirichlet boundary conditions of first kind

$$p(x,y,z) = g_0(x,y,z), \quad \frac{\partial p}{\partial n} = g_1(x,y,z), \quad (x,y,z) \in \partial\Omega, \tag{1.2}$$

or second kind

$$p(x,y,z) = g_0(x,y,z), \quad \frac{\partial^2 p}{\partial n^2} = g_2(x,y,z), \quad (x,y,z) \in \partial\Omega. \tag{1.3}$$

In 3D Cartesian coordinates, the biharmonic operator Δ^2 can be expressed as

$$\Delta^2 p(x,y,z) = \frac{\partial^4 p}{\partial x^4} + \frac{\partial^4 p}{\partial y^4} + \frac{\partial^4 p}{\partial z^4} + 2\frac{\partial^4 p}{\partial x^2 y^2} + 2\frac{\partial^4 p}{\partial x^2 z^2} + 2\frac{\partial^4 p}{\partial y^2 z^2}. \tag{1.4}$$

Numerous methods for numerically solving biharmonic equations have been proposed in the literature. One widely used technique is to split the equation $\Delta^2 p = f$ into two coupled Poisson equations for p and q: $\Delta p = q$ and $\Delta q = f$. This approach has been extensively used in previous work [2,6,7]. However, introducing the new variable q creates difficulties in defining its boundary conditions, and the accuracy of the numerical solution strongly depends on the method used to approximate these missing boundary values, which is the most challenging aspect of the coupling method. An alternative approach was proposed by Ribeiro Dos Santos in [5], which involves discretizing the equation on a uniform grid using a 25-point computational stencil with a second-order truncation error. This method requires modification at grid points near the boundaries, and it becomes extremely difficult to solve the resulting linear systems of 3D biharmonic equations by using direct methods when the number of unknowns is large. Iterative methods such as Jacobi or Gauss-Seidel can converge slowly or diverge, and the direct method is only applicable for moderate values of grid width h, according to Dehghan