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An Inverse Problem with the Final Overdetermination for the Mean Field Games System

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Abstract. The mean field games (MFG) theory has broad application in mathematical modeling of social phenomena. The Mean Field Games System (MFGS) is the key to the MFG theory. This is a system of two nonlinear parabolic partial differential equations with two opposite directions of time $t\in(0,T)$. The topic of Coefficient Inverse Problem (CIPs) for the MFGS is a newly emerging one. A CIP for the MFGS is studied. The input data are Dirichlet and Neumann boundary conditions either on a part of the lateral boundary (incomplete data) or on the whole lateral boundary (complete data). In addition to the initial conditions at $\{t=0\}$, terminal conditions at $\{t=T\}$ are given. The terminal conditions mean the final overdetermination. The necessity of assigning all these input data is explained. Hölder and Lipschitz stability estimates are obtained for the cases of incomplete and complete data respectively. These estimates imply uniqueness of the CIP.

AMS subject classifications: 35R30

Key words: The mean field games system, new Carleman estimates, Hölder and Lipschitz stability estimates, uniqueness.

1 Introduction

Social sciences play a significant role in the modern society. The Mean Field Games (MFG) theory studies the collective behavior of large populations of rational decision-makers (agents). This theory has a number of applications in the mathematical modeling

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of social phenomena. Some examples of these applications are, e.g., finance [1, 40], sociology [2], fighting corruption [23, 24], cyber security [24], etc.. Due to numerous applications of the MFG theory, it is important to study a variety of mathematical topics of this theory. In the current paper, we study a Coefficient Inverse Problem (CIP) for the Mean Field Games System (MFGS).

The MFG theory was first introduced in 2006-2007 in seminal works of Lasry and Lions [26–28] and of Huang, Caines and Malhamé [9, 10]. The MFGS is the core of the MFG theory. This is a system of two coupled nonlinear parabolic Partial Differential Equations (PDEs) with two opposite directions of time. In the first equation time is running downwards. This is the Hamilton-Jacobi-Bellman equation (HJB). And in the second equation time is running upwards. This is the Fokker-Planck (FP) equation. Let $\Omega\subset\mathbb{R}^n, n\geq 1$ be a bounded domain with its boundary $\partial\Omega$ and let time $t\in(0,T)$. The state position of a representative agent is $x\in\Omega$. The HJB equation governs the value function $u(x,t)$ of each individual agent located at x at the moment of time t. The FP equation describes the evolution of the distribution of agents $m(x,t)$ over time $t\in(0,T)$.

CIPs for the MFGS is a newly emerging topic. Uniqueness theorems for CIPs for the MFGS in the case of the input data resulting from infinitely many measurements were obtained in [**?**, 33, 35–37]. We refer to [5, 6] for numerical studies of CIPs for the MFGS. We consider the case of the input data resulting from a single measurement event [20, 21, 34]. We also refer to [7, 30, 32, 42] for some related studies. Previously a Hölder stability estimate was obtained in [21] for a CIP for the MFGS with a single measurement data. However, the statement of the CIP in [21] is significantly different from the one of this paper, see Subsection 3.1. In addition, Lipschitz stability estimate for a CIP with single measurement data was obtained in [12]. The MFGS of [12] does not contain the integral term, see Subsection 2.2 for this term.

Since the input data for CIPs are results of measurements, then they are given with errors. Hence, we are concerned with obtaining Hölder and Lipschitz stability estimates of the solution of our CIP with respect to the error in the input data. These stability estimates immediately imply uniqueness of our CIP.

In this paper we modify the framework, which was first proposed in [4], where the apparatus of Carleman estimates was introduced in the field of CIPs, also, see, e.g., [11, 13–17, 41] and references cited therein for some follow up publications. Carleman estimates were introduced in the MFG theory in [18] and were used since then in [12, 19, 21] as well as in the current paper. The idea of our modification of the framework of [4] is outlined in Subsection 3.2.

It is natural to call the problem of this paper "CIP with the final overdetermination". Indeed, we assume that we know both initial and terminal conditions for both functions u and m as well as both Dirichlet and Neumann boundary conditions for these functions on either a part of the lateral boundary or on the whole lateral boundary. On the other hand, if a CIP for a single parabolic equation requires to find a coefficient of this equation, assuming that its solution is known at $\{t=0\}$ and at $\{t=T\}$, then such a CIP is called "CIP with the final overdetermination": we refer to [13, Section 9.1] for the Lipschitz