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Pointwise Goal-Oriented a Posteriori Error Estimates Using Dual Problems with Dirac Delta Source Terms for Linear Elliptic Problems

Fei Li[1,2], Jingang Liu[4], Nianyu Yi[1,3,4] and Liuqiang Zhong[2,*]

[1] *School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China*

[2] *School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong 510631, China*

[3] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China*

[4] *Hunan National Center for Applied Mathematics, Xiangtan, Hunan 411105, China*

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Abstract. In this paper, a pointwise goal-oriented residual-based a posteriori error estimator is proposed for linear elliptic equations with restricted source terms. The pointwise error is directly estimated by introducing the dual problem with a Dirac delta source term instead of using classical mollification technique. The goal-oriented error estimator is proved to be the upper bound of the pointwise error. Numerical experiments show the advantage of the adaptive finite element method (AFEM) based on this error estimator, which can preserve the monotonicity of the pointwise error, compared with the goal-oriented AFEM using the mollification technique.

AMS subject classifications: 65M10, 78A48

Key words: Pointwise quantity of interest, a posteriori error estimate, adaptive finite element method, Dirac delta source term.

1 Introduction

Goal-oriented a posteriori error estimates have been widely attentioned in the past of more than two decades, and have been applied in many fields, such as elasticity [24, 30], hydromechanics [10, 16, 20], and electromagnetism [8, 21]. In contrast to a posteriori

*Corresponding author.
Email: lifeimath@scnu.edu.cn (F. Li), liujingang@xtu.edu.cn (J. Liu), yinianyu@xtu.edu.cn (N. Yi), zhong@scnu.edu.cn (L. Zhong)

error estimates [1, 31] that normally assess errors in the global energy norm, the goal-oriented a posteriori error estimates [5, 23, 26, 27, 32] evaluate errors of the specific goals, which are quantified as functional values of the solution of the problem. The functional values generally represent physical quantities of practical interest [4,6,15], and therefore, the goal-oriented a posteriori error estimate is usually more important in practice than the normal a posteriori error estimate. This paper focuses on the pointwise quantity of interest, i.e., the value of the solution at a point of interest in the physical domain, e.g., the temperature at a critical point, one can see [27–29] for the related work on this quantity of interest.

The quantity of interest is generally treated as an integral type of quantity of interest, which is an averaging of the solution over a small neighborhood of the point of interest, since the point value of the solution being a function in Sobolev spaces may not be well defined, i.e., the solution is discontinuous at the point of interest. This treatment is called mollification technique [27,28]. However, if the pointwise quantity of interest can be well defined, the goal-oriented a posteriori error estimate using the mollification technique does not assess the pointwise error but the error in another integral-averaging quantity of interest. Numerical experiments show that, for some model problems, this error estimate leads to that the pointwise error is not monotone-decreasing on the generating adaptive mesh sequence, which is unexpected (see Figs. 5 and 8 in Section 4).

The pointwise error estimation in this paper is different from the one using the mollification technique, and introduces a dual problem with the Dirac delta source term to directly estimate the pointwise error. We construct a theoretically reliable pointwise goal-oriented a posteriori error estimator, and numerical experiments indicate the advantage of the adaptive finite element method (AFEM) based on this error estimator that, for some model problems, it can preserve the monotone-decreasing behavior of the pointwise error, while the pointwise goal-oriented AFEM using the mollification technique cannot.

The structure of this paper is as follows: In Section 2, we present the primal model problem and the dual problem involved in their continuous, variational and discretization forms, and derive the error estimate in the pointwise quantity of interest. In Section 3, we define the primal, the dual, and the goal-oriented a posteriori error estimators, and prove their reliability. Section 3 ends up with formulating the adaptive finite element algorithm based on the pointwise goal-oriented error estimator. In Section 4, numerical experiments show the validity and advantage of the proposed algorithm.

2 Preliminaries

First, we introduce some notations with respect to spaces and norms. Given $D\subset\mathbb{R}^2$, let

$$L^r(D):=\{v\in L^1(D):\|v\|_{0,r,D}<\infty\},\quad 1\le r\le\infty,$$