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Meshfree Methods for the Helmholtz Equation with Variable Wave Speed

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Abstract. In this paper, we develop one-level and multilevel meshfree radial basis functions (RBF) collocation methods for solving the Helmholtz equation with variable wave speed on a bounded connected Lipschitz domain. The approximate solution is constructed by employing successive refinement scattered data sets and scaled compactly supported radial basis functions with varying support radii. We prove the convergence of one-level and multilevel collocation method for the modelling problem in Sobolev spaces. The convergence rates depend on the regularity of the solution, the smoothness of the computational domain, the bounds of frequency and wave speed, the approximation of scaled kernel-based spaces, the increasing rules of scattered data, and the selection of scaling parameters.

AMS subject classifications: 65N12, 65N15, 65N35

Key words: Helmholtz equation, radial basis functions, meshfree methods, collocation, variable wave speed.

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1 Introduction

The Helmholtz equation has many important applications, including time-harmonic acoustic wave-propagation, radar and sonar detection as well as medical and seismic imaging, and so on. A general Helmholtz equation in heterogeneous media has the form

$$-\operatorname{div}(a(\boldsymbol{x})\operatorname{grad} u)-\left(\frac{\omega}{c(\boldsymbol{x})}\right)^2 u=f(\boldsymbol{x}) \qquad \text{in } \Omega, \tag{1.1a}$$

$$a(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{n}}-i\omega\beta(\boldsymbol{x})u=g(\boldsymbol{x}) \qquad \text{on } \partial\Omega. \tag{1.1b}$$

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Where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, frequency $\omega \geq \omega_0 > 0$. For the coefficients, we suppose $a(\boldsymbol{x}), c(\boldsymbol{x})$ are bounded above and below by strictly positive numbers, real-valued $\beta(\boldsymbol{x}) \in L^\infty(\partial\Omega)$. It's usually assumed that $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ (or weaker assumption $g \in H^{-1/2}(\partial\Omega)$).

Several grid-type numerical methods have been designed to solve the Helmholtz equation, such as finite difference methods [3,10,11] and finite element methods [5,17,21,31]. To avoid mesh generation in complex domain, the radial basis functions has been gradually used for numerical solution of Helmholtz equation [22,26]. The radial basis functions allow the easy construction of approximation spaces in arbitrary dimensions with arbitrary smoothness. Therefore, this paper will consider one-level and multilevel radial basis collocation methods for the Helmholtz equation with variable wave speed.

The paper is organized as follows. Section 2 is devoted to some notation in Sobolev spaces. Section 3 discusses the well-posedness of Helmholtz equation and its priori bounds. Sections 4 and 5 provide the convergence of one-level collocation method and multilevel collocation method respectively. Section 6 gives several numerical examples and deals with numerical efficiency. The paper ends with some concluding remarks in Section 7.

2 Preliminaries

2.1 Sobolev space on Ω

When considering some complex-valued functions, we denote the scalar product and norm of $L^2(\Omega)$ as

$$(u,v) = \int_\Omega u\bar{v}d\boldsymbol{x}, \quad \|u\|_{L^2(\Omega)} = \left(\int_\Omega u\bar{u}d\boldsymbol{x}\right)^{1/2}, \quad u,v \in L^2(\Omega).$$

With a nonnegative integer k, Sobolev space $W^{k,2}(\Omega)$ contains all functions which have weak differentiability order k and integrability power 2. $W^{k,2}(\Omega)$ has been assembled with the (semi-) norms

$$|u|_{W^{k,2}(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}, \quad \|u\|_{W^{k,2}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

$W^{k,\infty}(\Omega)$ (semi-) norms are defined as

$$|u|_{W^{k,\infty}(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L^\infty(\Omega)}, \quad \|u\|_{W^{k,\infty}(\Omega)} = \sup_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Because $W^{k,2}(\Omega)$ is a Hilbert space, we usually denote $W^{k,2}(\Omega)$ as $H^k(\Omega)$. With a non-integer values $0 < \sigma < \infty$, the fractional order Sobolev space $H^\sigma(\mathbb{R}^d)$ is characterized by Fourier transform

$$\widehat{u}(\boldsymbol{\omega}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(\boldsymbol{x}) e^{-i\boldsymbol{x}^T\boldsymbol{\omega}} d\boldsymbol{x}. \tag{2.1}$$