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High-Efficiency Explicit Multistep Schemes for Coupled Second-Order FBSDEs

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Abstract. In this work, by introducing a new family of recursively defined processes, we propose new explicit multistep schemes for coupled second-order forward backward stochastic differential equations. The explicit schemes avoid calculating the conditional mathematical expectations of the generator f and calculate the required values of f explicitly and accurately. By combining the Sinc quadrature rule for approximating the conditional expectations, we further propose the kth order ($1 \leq k \leq 6$) fully discrete explicit multistep schemes. Numerical tests are presented to demonstrate the strong stability, high accuracy, and high efficiency of the explicit schemes.

AMS subject classifications: 65C20, 65C30, 60H35

Key words: Explicit multistep scheme, second-order forward backward stochastic differential equations, recursive approximation, Sinc quadrature rule.

1 Introduction

This paper is concerned with the numerical solution of the following second-order forward backward stochastic differential equations (2FBSDEs) on a filtered complete probability space $(\Omega,\mathcal{F},\mathbb{F},P)$:

$$\begin{cases} Y_t = \varphi(X_T) + \int_t^T f(s,\Theta_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \\ Z_t = Z_0 + \int_0^t A_s \mathrm{d}s + \int_0^t \Gamma_s \mathrm{d}W_s, \end{cases} \tag{1.1}$$

where $t \in [0,T]$ with the X_t being a certain diffusion process, $T > 0$ is the deterministic terminal time, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration generated by the standard q-dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$; $\varphi: \mathbb{R}^d \to \mathbb{R}$ is the terminal condition of the backward stochastic differential equation (BSDE); $f: [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{1\times q}\times\mathbb{R}^{q\times q}\to\mathbb{R}$

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is the generator of the BSDE; $\Theta_t = (X_t, Y_t, Z_t, \Gamma_t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times q} \times \mathbb{R}^{q\times q}$, $t \in [0,T]$, is the unknown stochastic process. An L^2-adapted solution of the 2FBSDEs (1.1) is a 5-tuple $(X_t, Y_t, Z_t, \Gamma_t, A_t)$, $t \in [0,T]$, which is $\mathcal{F}_t$-adapted, square integrable, and satisfies the 2FBSDEs (1.1).

The 2FBSDEs (1.1), as an extension of the BSDEs [2, 18], was first introduced in [5] and subsequently investigated further in [21]. The research of 2FBSDEs is driven by the connection between the 2FBSDEs and fully nonlinear partial differential equations (PDEs), specifically the Hamilton-Jacobi-Bellman equations and the Bellman-Isaacs equations which are extensively utilized in stochastic optimal control and stochastic differential games. This connection provides a stochastic representation for fully nonlinear PDEs, extending the nonlinear Feynman-Kac representations of linear and semi-linear parabolic PDEs (see, e.g., [15, 20] and references therein).

As the FBSDEs seldom admit explicitly closed-form solutions, numerical methods to BSDEs, FBSDEs and 2FBSDEs have played an important role in applications. Up to now, many numerical methods for BSDEs and FBSDEs have been proposed and analyzed [1, 3, 4, 6, 8, 13, 14, 16, 24, 25, 28, 30–32]. In the literature, the existing highly accurate numerical schemes rely on the high-order methods for both the forward and backward processes, and also require the values of the conditional mathematical expectations of the generator f. Furthermore, based on the local properties of the generator of diffusion processes and the Feynman-Kac formula, the authors in [29] presented implicit multistep schemes for coupled FBSDEs with high accuracy. The main features of the implicit schemes are that the forward SDE is solved using the Euler scheme, which dramatically reduces the computational complexity and enables the solution of complex problems with high accuracy.

Due to the complex solution structure, there are only few works on numerical methods for 2FBSDEs and fully nonlinear PDEs [7, 9, 11, 23, 27, 33]. In [9], a numerical scheme was proposed to solve the high-dimensional 2FBSDEs, and the numerical tests show that the scheme only achieves a low convergence rate. The authors in [33], proposed high-order multistep schemes for 2FBSDEs, to implicitly solve the solution accurately with the forward SDE solved by the Euler scheme. Thus some iterations are needed for the implicit solving processes, which may affect the efficiency of the schemes, especially for solving coupled 2FBSDEs. The other existing works require high-order methods for the forward process to achieve high accuracy.

In order to design a highly accurate and highly efficient scheme for 2FBSDEs, we propose new explicit multistep schemes for solving 2FBSDEs in this paper. The main contributions of this paper are outlined as follows.

- We first introduce a new family of recursively defined processes, then propose new explicit multistep schemes for 2FBSDEs. The main features of the schemes are that the solutions $(Y^n, Z^n, \Gamma^n, A^n)$ of the schemes are solved explicitly and accurately, and only the values of $\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+i}}]$, $\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+i}}(\Delta W_{n,i})^\top]$, $\mathbb{E}_{t_n}^{X^n}[Z_{t_{n+i}}]$ and $\mathbb{E}_{t_n}^{X^n}[Z_{t_{n+i}}^\top(\Delta W_{n,i})^\top]$ $(i = 1, \cdots, k)$ are required to explicitly calculate the generator f, the forward SDE is