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High-Efficiency Explicit Multistep Schemes for Coupled Second-Order FBSDEs

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Abstract. In this work, by introducing a new family of recursively defined processes, we propose new explicit multistep schemes for coupled second-order forward backward stochastic differential equations. The explicit schemes avoid calculating the conditional mathematical expectations of the generator f and calculate the required values of f explicitly and accurately. By combining the Sinc quadrature rule for approximating the conditional expectations, we further propose the kth order ($1\leq k\leq 6$) fully discrete explicit multistep schemes. Numerical tests are presented to demonstrate the strong stability, high accuracy, and high efficiency of the explicit schemes.

AMS subject classifications: 65C20, 65C30, 60H35

Key words: Explicit multistep scheme, second-order forward backward stochastic differential equations, recursive approximation, Sinc quadrature rule.

1 Introduction

This paper is concerned with the numerical solution of the following second-order forward backward stochastic differential equations (2FBSDEs) on a filtered complete probability space $(\Omega,\mathcal{F},\mathbb{F},P)$:

$$\begin{cases} Y_t=\varphi(X_T)+\displaystyle\int_t^T f(s,\Theta_s)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s, \\ Z_t=Z_0+\displaystyle\int_0^t A_s\mathrm{d}s+\int_0^t \Gamma_s\mathrm{d}W_s, \end{cases} \tag{1.1}$$

where $t\in[0,T]$ with the X_t being a certain diffusion process, $T>0$ is the deterministic terminal time, $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration generated by the standard q-dimensional Brownian motion $W=(W_t)_{0\leq t\leq T}$; $\varphi:\mathbb{R}^d\to\mathbb{R}$ is the terminal condition of the backward stochastic differential equation (BSDE); $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{1\times q}\times\mathbb{R}^{q\times q}\to\mathbb{R}$

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is the generator of the BSDE; $\Theta_t = (X_t, Y_t, Z_t, \Gamma_t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{1\times q} \times \mathbb{R}^{q\times q}$, $t \in [0,T]$, is the unknown stochastic process. An L^2-adapted solution of the 2FBSDEs (1.1) is a 5-tuple $(X_t, Y_t, Z_t, \Gamma_t, A_t)$, $t\in[0,T]$, which is $\mathcal{F}_t$-adapted, square integrable, and satisfies the 2FBSDEs (1.1).

The 2FBSDEs (1.1), as an extension of the BSDEs [2, 18], was first introduced in [5] and subsequently investigated further in [21]. The research of 2FBSDEs is driven by the connection between the 2FBSDEs and fully nonlinear partial differential equations (PDEs), specifically the Hamilton-Jacobi-Bellman equations and the Bellman-Isaacs equations which are extensively utilized in stochastic optimal control and stochastic differential games. This connection provides a stochastic representation for fully nonlinear PDEs, extending the nonlinear Feynman-Kac representations of linear and semi-linear parabolic PDEs (see, e.g., [15, 20] and references therein).

As the FBSDEs seldom admit explicitly closed-form solutions, numerical methods to BSDEs, FBSDEs and 2FBSDEs have played an important role in applications. Up to now, many numerical methods for BSDEs and FBSDEs have been proposed and analyzed [1,3,4,6,8,13,14,16,24,25,28,30–32]. In the literature, the existing highly accurate numerical schemes rely on the high-order methods for both the forward and backward processes, and also require the values of the conditional mathematical expectations of the generator f. Furthermore, based on the local properties of the generator of diffusion processes and the Feynman-Kac formula, the authors in [29] presented implicit multistep schemes for coupled FBSDEs with high accuracy. The main features of the implicit schemes are that the forward SDE is solved using the Euler scheme, which dramatically reduces the computational complexity and enables the solution of complex problems with high accuracy.

Due to the complex solution structure, there are only few works on numerical methods for 2FBSDEs and fully nonlinear PDEs [7,9,11,23,27,33]. In [9], a numerical scheme was proposed to solve the high-dimensional 2FBSDEs, and the numerical tests show that the scheme only achieves a low convergence rate. The authors in [33], proposed high-order multistep schemes for 2FBSDEs, to implicitly solve the solution accurately with the forward SDE solved by the Euler scheme. Thus some iterations are needed for the implicit solving processes, which may affect the efficiency of the schemes, especially for solving coupled 2FBSDEs. The other existing works require high-order methods for the forward process to achieve high accuracy.

In order to design a highly accurate and highly efficient scheme for 2FBSDEs, we propose new explicit multistep schemes for solving 2FBSDEs in this paper. The main contributions of this paper are outlined as follows.

- We first introduce a new family of recursively defined processes, then propose new explicit multistep schemes for 2FBSDEs. The main features of the schemes are that the solutions $(Y^n, Z^n, \Gamma^n, A^n)$ of the schemes are solved explicitly and accurately, and only the values of $\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+i}}]$, $\mathbb{E}_{t_n}^{X^n}[Y_{t_{n+i}}(\Delta W_{n,i})^\top]$, $\mathbb{E}_{t_n}^{X^n}[Z_{t_{n+i}}]$ and $\mathbb{E}_{t_n}^{X^n}[Z_{t_{n+i}}^\top(\Delta W_{n,i})^\top]$ $(i=1,\cdots,k)$ are required to explicitly calculate the generator f, the forward SDE is solved by the Euler method, which makes the schemes simple in use, and solving (Y^n,Z^n,Γ^n,A^n) can be fully parallelized for different X^n in space.

- Our numerical tests show the high capacity of our new explicit schemes for solving complex problems, such as decoupled and coupled 2FBSDEs and fully nonlinear HJB equations.

The rest of this paper is organized as follows. In Section 2, we give the discretization procedure of 2FBSDEs by introducing a new family of recursively defined processes. Our explicit multistep schemes for solving 2FBSDEs are given in Section 3. In Section 4, several numerical tests are presented to demonstrate the strong stability, high accuracy and high efficiency of our explicit multistep schemes. We give some conclusions in Section 5.

2 Discretizations

For the time interval $[0,T]$, we introduce the time partition

$$0=t_0<t_1<t_2<\cdots<t_N=T, \tag{2.1}$$

where N is a positive integer. For this partition, we set $\Delta t_n=t_{n+1}-t_n$, $\Delta t_{n,j}=t_{n+j}-t_n$, $\Delta t_{t_n,t}=t-t_n$ for $t\geq t_n$, and $\Delta t=\max_{0\leq n\leq N-1}\Delta t_n$. We denote $\Delta W_{n+1}=W_{t_{n+1}}-W_{t_n}$, by $\Delta W_{n,j}$ the increment $W_{t_{n+j}}-W_{t_n}$, and $\Delta W_{t_n,t}=W_t-W_{t_n}$. For uniform time partition, it holds that $\Delta t_n=\Delta t=\frac{T}{N}$.

2.1 Time discretization of 2FBSDEs

Let $\Theta_t=(X_t,Y_t,Z_t,\Gamma_t)$ be the solution of the decoupled 2FBSDEs (1.1), where the forward diffusion process X_t is defined as

$$X_t=X_0+\int_0^t b(s,X_s)\mathrm{d}s+\int_0^t \sigma(s,X_s)\mathrm{d}W_s,\quad t\in[0,T], \tag{2.2}$$

with drift coefficient $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and diffusion coefficient $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times q}$. Then we have the equations

$$Y_{t_n}=Y_t+\int_{t_n}^t f(s,\Theta_s)\mathrm{d}s-\int_{t_n}^t Z_s\mathrm{d}W_s,\qquad t\in[t_n,T], \tag{2.3a}$$

$$Z_t=Z_{t_n}+\int_{t_n}^t A_s\mathrm{d}s+\int_{t_n}^t \Gamma_s\mathrm{d}W_s,\qquad t\in[t_n,T]. \tag{2.3b}$$

Under certain conditions on the generator f, by taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]=\mathbb{E}[\cdot|\mathcal{F}_{t_n},X_{t_n}=x]$ on both sides of (2.3a) and (2.3b), we get

$$Y_{t_n}=\mathbb{E}_{t_n}^x[Y_t]+\int_{t_n}^t \mathbb{E}_{t_n}^x[f(s,\Theta_s)]\mathrm{d}s,\qquad t\in[t_n,T], \tag{2.4a}$$

$$Z_{t_n}=\mathbb{E}_{t_n}^x[Z_t]-\int_{t_n}^t \mathbb{E}_{t_n}^x[A_s]\mathrm{d}s,\qquad t\in[t_n,T]. \tag{2.4b}$$

By the Feynman-Kac formula [5, 20, 21], the conditional expectation $\mathbb{E}_{t_n}^x[f(s,\Theta_s)]$ and $\mathbb{E}_{t_n}^x[A_s]$ are continuous functions of s under the filtration $\mathcal{F}_t$. Thus by taking the derivatives with respect to t on both sides of (2.4a) and (2.4b), we derive the following two reference ordinary differential equations (ODEs)

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Y_t]}{\mathrm{d}t} = -\mathbb{E}_{t_n}^x[f(t,\Theta_t)], \qquad t\in(t_n,T], \tag{2.5a}$$

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Z_t]}{\mathrm{d}t} = \mathbb{E}_{t_n}^x[A_t], \qquad t\in(t_n,T]. \tag{2.5b}$$

Multiplying both sides of (2.3a) and (2.3b) by $(\Delta W_{t_n,t})^\top$, and taking the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the derived equations, for $t\in[t_n,T]$, we obtain

$$0=\mathbb{E}_{t_n}^x[Y_t(\Delta W_{t_n,t})^\top]+\int_{t_n}^t \mathbb{E}_{t_n}^x[f(s,\Theta_s)(\Delta W_{t_n,s})^\top]\mathrm{d}s-\int_{t_n}^t \mathbb{E}_{t_n}^x[Z_s]\mathrm{d}s, \tag{2.6a}$$

$$0=\mathbb{E}_{t_n}^x[Z_t^\top(\Delta W_{t_n,t})^\top]-\int_{t_n}^t \mathbb{E}_{t_n}^x[A_s^\top(\Delta W_{t_n,s})^\top]\mathrm{d}s-\int_{t_n}^t \mathbb{E}_{t_n}^x[\Gamma_s]\mathrm{d}s. \tag{2.6b}$$

Here the Itô isometry formula is used. Upon taking the derivatives with respect to $t\in(t_n,T)$ in (2.6a) and (2.6b), one gets the ODEs

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Y_t(\Delta W_{t_n,t})^\top]}{\mathrm{d}t} = -\mathbb{E}_{t_n}^x[f(t,\Theta_t)(\Delta W_{t_n,t})^\top]+\mathbb{E}_{t_n}^x[Z_t], \tag{2.7a}$$

$$\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Z_t^\top(\Delta W_{t_n,t})^\top]}{\mathrm{d}t} = \mathbb{E}_{t_n}^x[A_t^\top(\Delta W_{t_n,t})^\top]+\mathbb{E}_{t_n}^x[\Gamma_t]. \tag{2.7b}$$

Now let $(\bar{Y}_t,\bar{Z}_t)=(Y_t(\bar{X}_t^{t_n,x}),Z_t(\bar{X}_t^{t_n,x}))$ with the $\bar{X}_t^{t_n,x}$ defined by

$$\bar{X}_t^{t_n,x}=x+\int_{t_n}^t \bar{b}(s,\bar{X}_s^{t_n,x})\mathrm{d}s+\int_{t_n}^t \bar{\sigma}(s,\bar{X}_s^{t_n,x})\mathrm{d}W_s, \tag{2.8}$$

where the coefficients $\bar{b}(t_n,x)=b(t_n,x)$ and $\bar{\sigma}(t_n,x)=\sigma(t_n,x)$. By the local properties of the generator of diffusion process [29], we have the following identities

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Y_t]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t]}{\mathrm{d}t}\right|_{t=t_n},\\
\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Z_t]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t]}{\mathrm{d}t}\right|_{t=t_n},\\
\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Y_t(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n},\\
\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[Z_t^\top(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n} &= \left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t^\top(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n}.
\end{aligned}$$

By the facts

$$\begin{aligned}
&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[f(t,\Theta_t)]=f(t_n,\Theta_{t_n}), &&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[A_t]=A_{t_n},\\
&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[Z_t]=Z_{t_n}, &&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[\Gamma_t]=\Gamma_{t_n},\\
&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[A_t^\top(\Delta W_{t_n,t})^\top]=0, &&\lim_{t\to t_n+0}\mathbb{E}_{t_n}^x[f(t,\Theta_t)(\Delta W_{t_n,t})^\top]=0,
\end{aligned}$$

and letting $t\to t_n+0$ in the above four equations (2.5a), (2.5b), (2.7a) and (2.7b), we deduce

$$\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t]}{\mathrm{d}t}\right|_{t=t_n+0}=-f(t_n,\Theta_{t_n}), \tag{2.9a}$$

$$\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t]}{\mathrm{d}t}\right|_{t=t_n+0}=A_{t_n}, \tag{2.9b}$$

$$\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n+0}=Z_{t_n}^\top, \tag{2.9c}$$

$$\left.\frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t^\top(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\right|_{t=t_n+0}=\Gamma_{t_n}. \tag{2.9d}$$

Now approximating the derivatives in Eqs. (2.9a)-(2.9d) by $\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}]$, $\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}]$, $\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}(\Delta W_{n,j})^\top]$ and $\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}^\top(\Delta W_{n,j})^\top]$ leads to

$$\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}]=-f(t_n,X_{t_n},Y_{t_n},Z_{t_n},\Gamma_{t_n})+R_{y,k}^n, \tag{2.10a}$$

$$\sum_{j=0}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}]=A_{t_n}+R_{A,k}^n, \tag{2.10b}$$

$$\sum_{j=1}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}(\Delta W_{n,j})^\top]=Z_{t_n}^\top+R_{z,k}^n, \tag{2.10c}$$

$$\sum_{j=1}^k\alpha_{k,j}^n\mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}^\top(\Delta W_{n,j})^\top]=\Gamma_{t_n}+R_{\Gamma,k}^n, \tag{2.10d}$$

where the $\alpha_{k,j}^n$, $j=0,1,\cdots,k$, are defined by the linear algebraic equations

$$\sum_{j=0}^k\alpha_{k,j}^n(\Delta t_{n,j})^i=\delta_{i1},\quad i=0,1,\cdots,k, \tag{2.11}$$

Table 1: The coefficients $\alpha_{k,j}^n \Delta t$ for $k=1,2,\cdots,6$.

$\alpha_{k,j}^n \Delta t$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$k=1$	-1	1					
$k=2$	$-\frac{3}{2}$	2	$-\frac{1}{2}$				
$k=3$	$-\frac{11}{6}$	3	$-\frac{3}{2}$	$\frac{1}{3}$			
$k=4$	$-\frac{25}{12}$	4	-3	$\frac{4}{3}$	$-\frac{1}{4}$		
$k=5$	$-\frac{137}{60}$	5	-5	$\frac{10}{3}$	$-\frac{5}{4}$	$\frac{1}{5}$	
$k=6$	$-\frac{49}{20}$	6	$-\frac{15}{2}$	$\frac{20}{3}$	$-\frac{15}{4}$	$\frac{6}{5}$	$-\frac{1}{6}$

with δ_{ij} is the Kronecker δ function, and the four terms $R_{y,k}^n$, $R_{A,k}^n$, $R_{z,k}^n$ and $R_{\Gamma,k}^n$ are the truncation errors,

$$
\begin{aligned}
R_{y,k}^n &= \sum_{j=0}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}] - \frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t]}{\mathrm{d}t}\bigg|_{t=t_n+0}, \\
R_{z,k}^n &= \sum_{j=1}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}(\Delta W_{n,j})^\top] - \frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Y}_t(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\bigg|_{t=t_n+0}, \\
R_{A,k}^n &= \sum_{j=0}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}] - \frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t]}{\mathrm{d}t}\bigg|_{t=t_n+0}, \\
R_{\Gamma,k}^n &= \sum_{j=1}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Z}_{t_{n+j}}^\top(\Delta W_{n,j})^\top] - \frac{\mathrm{d}\mathbb{E}_{t_n}^x[\bar{Z}_t^\top(\Delta W_{t_n,t})^\top]}{\mathrm{d}t}\bigg|_{t=t_n+0}.
\end{aligned}
$$

It should be noted that the above algebraic equations are unstable for $k \geq 7$ by the theory of ODEs [29,33]. Specifically, when $\Delta t_{n,j} = j\Delta t$, $\alpha_{k,j}^n \Delta t$ $(j=0,1,\cdots,k)$ for $1 \leq k \leq 6$ are listed in Table 1. Under certain conditions, the functions $\mathbb{E}_{t_n}^x[Y_t(\Delta W_{t_n,t})^\top]$, $\mathbb{E}_{t_n}^x[Z_t^\top(\Delta W_{t_n,t})^\top]$, $\mathbb{E}_{t_n}^x[Y_t]$, $\mathbb{E}_{t_n}^x[Z_t]$ and their derivatives with respect to t are bounded. In this case, we have the estimates

$$
R_{y,k}^n = \mathcal{O}((\Delta t)^k), \quad R_{z,k}^n = \mathcal{O}((\Delta t)^k), \quad R_{A,k}^n = \mathcal{O}((\Delta t)^k), \quad R_{\Gamma,k}^n = \mathcal{O}((\Delta t)^k). \tag{2.12}
$$

2.2 Recursive approximations

Based on the four discrete equations (2.10a)-(2.10d), the authors in [33] approximated the Y_{t_n} implicitly and proposed an effective high-order implicit multistep scheme for solving 2FBSDEs. Numerical tests showed that the scheme can solve 2FBSDEs effectively with high accuracy. However, the efficiency of the scheme in [33] may be inefficient when using it to solve 2FBSDEs.

To improve the efficiency, in this paper, we will propose an explicit multistep scheme for solving 2FBSDEs with high accuracy and efficiency. To this end, with $Y_{t_n,0} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}]$, we first introduce a family of processes $Y_{t_n,i}$, $i=1,2,\cdots,k-1$, recursively defined by

$$\alpha_{i,0}^n Y_{t_n,i} = -\sum_{j=1}^{i} \alpha_{i,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}] - f(t_n, X_{t_n}, Y_{t_n,i-1}, Z_{t_n}, \Gamma_{t_n}). \tag{2.13}$$

Then we rewrite (2.10a) in the following form

$$\sum_{j=0}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}] = -f(t_n, X_{t_n}, Y_{t_n,k-1}, Z_{t_n}, \Gamma_{t_n}) + \bar{R}_{y,k}^n, \tag{2.14}$$

where

$$\bar{R}_{y,k}^n = R_{y,k}^n + (f(t_n, X_{t_n}, Y_{t_n,k-1}, Z_{t_n}, \Gamma_{t_n}) - f(t_n, X_{t_n}, Y_{t_n}, Z_{t_n}, \Gamma_{t_n})),$$

and $Y_{t_n,k-1}$ is the recursive approximation of Y_{t_n} defined by (2.13).

About the recursive approximations $Y_{t_n,i}$, $i=1,2,\cdots,k-1$, and the truncation error term $\bar{R}_{y,k}^n$, we have the following lemma.

Lemma 2.1. *Let $\bar{R}_{y,k}^n$ be defined in* (2.14) *and $Y_{t_n,i}$, $i=1,2,\cdots,k-1$, be defined by* (2.13). *Then under appropriate regularity conditions on the b, σ, f and φ, it holds*

$$\bar{R}_{y,k}^n = \mathcal{O}((\Delta t)^k), \tag{2.15a}$$

$$Y_{t_n,i} = Y_{t_n} + \tilde{R}_{y,i}^n, \tag{2.15b}$$

with $\tilde{R}_{y,i}^n = \mathcal{O}((\Delta t)^{i+1})$ for $i=1,2,\cdots,k-1$.

Proof. By the nonlinear Feynman-Kac formula [19,20] and Itô's formula, we have

$$\mathbb{E}_{t_n}^x[Y_{t_{n+1}}] = Y_{t_n} + \mathcal{O}(\Delta t),$$

which implies that (2.15b) is true for $i=0$. Assume that the estimates (2.15b) hold true for $i=1,\cdots,k-1$. Then by (2.10a) and (2.13), we deduce

$$\begin{aligned}
\alpha_{k,0}^n Y_{t_n,k} &= -\sum_{j=1}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}] - f(t_n, X_{t_n}, Y_{t_n} + \mathcal{O}((\Delta t)^k), Z_{t_n}, \Gamma_{t_n}) \\
&= -\sum_{j=1}^{k} \alpha_{k,j}^n \mathbb{E}_{t_n}^x[\bar{Y}_{t_{n+j}}] - f(t_n, X_{t_n}, Y_{t_n}, Z_{t_n}, \Gamma_{t_n}) + \mathcal{O}((\Delta t)^k) \\
&= \alpha_{k,0}^n Y_{t_n} - R_{y,k}^n + \mathcal{O}((\Delta t)^k) \\
&= \alpha_{k,0}^n Y_{t_n} + \mathcal{O}((\Delta t)^k).
\end{aligned} \tag{2.16}$$

By multiplying both sides of the above equation by Δt_n, we deduce

$$\alpha_{k,0}^n \Delta t_n Y_{t_n,k} = \alpha_{k,0}^n \Delta t_n Y_{t_n} + \mathcal{O}((\Delta t)^{k+1}), \tag{2.17}$$

which implies the estimate (2.15b) holds true for $i=k$. Thus by induction, we prove that the estimates (2.15b) hold true for $i>0$. By (2.15b) and the definitions of $R^n_{y,k}$ and $\bar{R}^n_{y,k}$ in (2.10a) and (2.14), respectively, we derive

$$\begin{aligned}\bar{R}^n_{y,k}=&R^n_{y,k}+(f(t_n,X_{t_n},Y_{t_n,k-1},Z_{t_n},\Gamma_{t_n})-f(t_n,X_{t_n},Y_{t_n},Z_{t_n},\Gamma_{t_n}))\\=&R^n_{y,k}+\mathcal{O}(\tilde{R}^n_{y,k-1})=\mathcal{O}((\Delta t)^k),\end{aligned} \tag{2.18}$$

which implies that (2.15a) holds. The proof ends. □

Now Eqs. (2.14) and (2.10b)-(2.10d) are our discretizations for solving the decoupled 2FBSDEs (1.1), which will be used to derive our numerical schemes.

3 Numerical schemes for 2FBSDEs

3.1 The time semi-discrete scheme

Based on the discretizations (2.14) and (2.10b)-(2.10d), we will present our new explicit multistep schemes for solving 2FBSDEs. Let X^n, Y^n, Z^n, A^n and Γ^n be the numerical approximations of the solutions X_t, Y_t, Z_t, A_t and Γ_t of the decoupled 2FBSDEs in (1.1) at time t_n, respectively. For $t\in[t_n,T]$, we use the simplest choice of $\bar{b}$ and $\bar{\sigma}$ defined as $\bar{b}(t,\bar{X}^{t_n,x}_t)=b(t_n,x)$ and $\bar{\sigma}(t,\bar{X}^{t_n,x}_t)=\sigma(t_n,x)$. By removing the truncation error terms $\bar{R}^n_{y,k}$, $R^n_{A,k}$, $R^n_{z,k}$ and $R^n_{\Gamma,k}$ in (2.14) and (2.10b)-(2.10d), we propose the following new time semi-discrete explicit multistep Scheme 3.1 for decoupled 2FBSDEs.

Scheme 3.1. Given Y^{N-j} and Z^{N-j}, $j=0,\cdots,k-1$, for $n=N-k,\cdots,0$, solve $\bar{X}^{n,j}$ $(j=1,\cdots,k)$, $Y^n=Y^n(X^n)$, $Z^n=Z^n(X^n)$, $A^n=A^n(X^n)$ and $\Gamma^n=\Gamma^n(X^n)$ for $X^n\in\mathbb{R}^d$ by

$$\bar{X}^{n,j}=X^n+b(t_n,X^n)\Delta t_{n,j}+\sigma(t_n,X^n)\Delta W_{n,j},\qquad j=1,\cdots,k, \tag{3.1a}$$

$$Z^n=\sum_{j=1}^k\alpha^n_{k,j}\mathbb{E}^{X^n}_{t_n}[\bar{Y}^{n+j}(\Delta W_{n,j})^\top], \tag{3.1b}$$

$$A^n=\sum_{j=0}^k\alpha^n_{k,j}\mathbb{E}^{X^n}_{t_n}[\bar{Z}^{n+j}], \tag{3.1c}$$

$$\Gamma^n=\sum_{j=1}^k\alpha^n_{k,j}\mathbb{E}^{X^n}_{t_n}[(\bar{Z}^{n+j})^\top(\Delta W_{n,j})^\top], \tag{3.1d}$$

$$\sum_{j=0}^k\alpha^n_{k,j}\mathbb{E}^{X^n}_{t_n}[\bar{Y}^{n+j}]=-f(t_n,X^n,Y^n_{k-1},Z^n,\Gamma^n), \tag{3.1e}$$

with $\bar{Y}^{n+j}=Y^{n+j}(\bar{X}^{n,j})$, $\bar{Z}^{n+j}=Z^{n+j}(\bar{X}^{n,j})$, and Y^n_{k-1} is recursively calculated by

$$\alpha^n_{i,0}Y^n_i=-\sum_{j=1}^i\alpha^n_{i,j}\mathbb{E}^{X^n}_{t_n}[\bar{Y}^{n+j}]-f(t_n,X^n,Y^n_{i-1},Z^n,\Gamma^n),\quad i=1,2,\cdots,k-1, \tag{3.2}$$

with $Y_0^n = \mathbb{E}_{t_n}^{X^n}[\bar{Y}^{n+1}]$.

Remark 3.1. The local truncation errors of Scheme 3.1 are $\bar{R}^n_{y,k}$, $R^n_{A,k}$, $R^n_{z,k}$ and $R^n_{\Gamma,k}$ defined in (2.14) and (2.10b)-(2.10d), respectively. Under certain conditions on φ, b, σ and f, the following estimates hold.

$$\bar{R}^n_{y,k} = \mathcal{O}((\Delta t)^k), \qquad R^n_{z,k} = \mathcal{O}((\Delta t)^k), \tag{3.3a}$$

$$R^n_{A,k} = \mathcal{O}((\Delta t)^k), \qquad R^n_{\Gamma,k} = \mathcal{O}((\Delta t)^k). \tag{3.3b}$$

3.2 The time-space fully discrete scheme

Now we present our new time-space fully discrete scheme for the decoupled 2FBSDEs (1.1) by approximating the expectations using the Sinc quadrature rule. To this end, we introduce a spatial partition $\mathcal{D}_h^d = \{x_j = (x^1_{j_1}, x^2_{j_2}, \cdots, x^d_{j_d})^\top | x_j \in \mathbb{R}^d,\ j_m \in \mathbb{Z},\ m = 1, \cdots, d\}$ for space $\mathbb{R}^d$, where $\mathbb{Z}$ is the set of integer numbers and Δx^m is the uniform space step in the m-th direction, that is, $x^m_k = k\Delta x^m$.

3.2.1 Sinc approximations

First, we recall some fundamental concepts of the Sinc approximations [12, 22]. The Sinc function is defined on the real line $\mathbb{R}$ by

$$\mathrm{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{3.4}$$

For a function g defined on $\mathbb{R}$, its Cardinal function $C(g,h)$ is defined by

$$C(g,h)(x) = \sum_{k=-\infty}^{\infty} g(kh)\mathrm{sinc}\Big(\frac{x-kh}{h}\Big), \quad \forall x \in \mathbb{R}, \tag{3.5}$$

where h is a positive real number. To investigate the convergence of the series in (3.5), the function class $B(h)$ is defined as the set of entire functions g such that $g \in L^2(\mathbb{R})$ on the real line $\mathbb{R}$, and g has the following exponential type in the entire complex plane $\mathbb{C}$

$$|g(z)| \leq K \exp\Big(\frac{\pi|z|}{h}\Big), \quad \forall z \in \mathbb{C}, \tag{3.6}$$

where K and h are positive constants. Then for function $g \in B(h)$, we have

$$g(z) = \sum_{k=-\infty}^{\infty} g(kh)\mathrm{Sinc}\Big(\frac{z-kh}{h}\Big), \quad \forall z \in \mathbb{C}. \tag{3.7}$$

Furthermore, if $\sum_{k=-\infty}^{\infty} g(kh)$ converges, for sufficiently small h, it holds

$$\int_{-\infty}^{\infty} g(x)\mathrm{d}x = h \sum_{k=-\infty}^{\infty} g(kh). \tag{3.8}$$

Given a positive integer M, by $T_M(g,h)$ we denote the truncated Sinc approximation

$$T_M(g,h)=h\sum_{k=-M}^{M}g(kh), \tag{3.9}$$

which is also called the Sinc quadrature rule for computing $\int_{\mathbb{R}}g(x)\mathrm{d}x$ [25]. Additionally, by $\eta_M(g,h)$ we denote the truncation error of $T_M(g,h)$, that is,

$$\eta_M(g,h)=\int_{\mathbb{R}}g(x)\mathrm{d}x-T_M(g,h). \tag{3.10}$$

Assume the function g is bounded, for sufficiently small h, if there exists a positive number γ such that $\gamma\leq Mh^2$, then we have

$$|\eta_M(g,h)|\leq Ch\exp\Big(-\frac{M^2h^2}{2}\Big), \tag{3.11}$$

where C is a positive constant depending on γ and the upper bound of g. For more details about the Sinc quadrature rule, readers may refer to [25].

3.2.2 The fully discrete scheme

With the above spatial partition and the Sinc quadrature rule, we aim to solve the solution (Y^n,Z^n,Γ^n,A^n) at grid point $x\in\mathcal{D}_h^d$. In Scheme 3.1, the conditional mathematical expectations in the form of $\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]$ are needed to be calculated for properly defined functions $g:\mathbb{R}^d\to\mathbb{R}$, where

$$\bar{X}^{n,j}=x+b_{t_n}\Delta t_{n,j}+\sigma_{t_n}\Delta W_{n,j}$$

with $b_{t_n}=b(t_n,x)$ and $\sigma_{t_n}=\sigma(t_n,x)$. Noting that

$$\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]=\frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d}g\Big(x+b_{t_n}\Delta t_{n,j}+\sigma_{t_n}\sqrt{\Delta t_{n,j}}s\Big)\exp\Big(-\frac{s^\top s}{2}\Big)\mathrm{d}s, \tag{3.12}$$

where $s=(s_1,\cdots,s_d)^\top$, by (3.8), we have

$$\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]=\frac{\det(H)}{(\sqrt{2\pi})^d}\sum_{k=-\infty}^{\infty}g(x+b_{t_n}\Delta t_{n,j}+\sigma_{t_n}\sqrt{\Delta t_{n,j}}Hk)\exp\Big(-\frac{(Hk)^\top Hk}{2}\Big), \tag{3.13}$$

where $k=(k_1,\cdots,k_d)^\top$, H is the diagonal matrix $\mathrm{Diag}(h_1,\cdots,h_d)$ with the parameters $\{h_i\}_{i=1}^d$ in the Sinc quadrature rule, $\sum_{k=-\infty}^{\infty}=\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_d=-\infty}^{\infty}$, and $\det(H)$ represents the determinant of the square matrix H.

Generally, $x+b_{t_n}\Delta t_{n,j}+\sigma_{t_n}\sqrt{\Delta t_{n,j}}Hk$ in (3.13) does not belong to $\mathcal{D}_h^d$. Thus the values of $g(x+\sqrt{\Delta t_{n,j}}Hk)$ should be approximated by using the values of the information of g

on $\mathcal{D}_h^d$, which leads to spatial interpolations. To avoid spatial interpolations, according to the authors in [25], we define the transformation $\phi^{n,j}$ as

$$\tau=\phi^{n,j}(s)=b_{t_n}\sqrt{\Delta t_{n,j}}+\sigma_{t_n}s, \tag{3.14}$$

which inverse transformation is

$$s=\psi^{n,j}(\tau)=(\sigma_{t_n})^{-1}(\tau-b_{t_n}\sqrt{\Delta t_{n,j}}).$$

Using these two transformations, (3.13) becomes

$$\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]=\frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d}g(x+\sqrt{\Delta t_{n,j}}\tau)\exp\Big(-\frac{\|\psi^{n,j}(\tau)\|^2}{2}\Big)\frac{1}{|\det(\sigma_{t_n})|}\mathrm{d}\tau, \tag{3.15}$$

where $\|\cdot\|$ represents the Euclidean norm. Let i be the index such that $x=x_i\in\mathcal{D}_h^d$. By imposing the requirement on the parameter matrix H as

$$H=\mathrm{Diag}(h_1,\cdots,h_d)=R\mathrm{Diag}\Big(\frac{\Delta x^1}{\sqrt{\Delta t_{n,j}}},\cdots,\frac{\Delta x^d}{\sqrt{\Delta t_{n,j}}}\Big), \tag{3.16}$$

where $R=\mathrm{Diag}(r_1,\cdots,r_d)$ with $\{r_i\}_{i=1}^d$ being positive integers. By applying the Sinc quadrature rule (3.9), we get

$$\begin{aligned}\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]&=\sum_{k=-M}^{M}v_k^{i,n,j}g(x_{i+Rk})+\Upsilon_{g,M}^{n,j}\\&=\sum_{k=-M}^{M}\lambda_k^{i,n,j,M}g(x_{i+Rk})+\hat{\Upsilon}_{g,M}^{n,j},\end{aligned} \tag{3.17}$$

where

$$\lambda_k^{i,n,j,M}=\frac{v_k^{i,n,j}}{\sum_{k=-M}^{M}v_k^{i,n,j}}\quad\text{with}\quad v_k^{i,n,j}=\frac{|\det(H(\sigma_{t_n})^{-1})|}{\sqrt{(2\pi)^d}}\exp\Big(-\frac{\|\psi^{n,j}(H\boldsymbol{k})\|^2}{2}\Big),$$

$\Upsilon_{g,M}^{n,j}$ and $\hat{\Upsilon}_{g,M}^{n,j}$ are the Sinc quadrature truncation errors

$$\begin{aligned}\Upsilon_{g,M}^{n,j}&=\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]-\sum_{k=-M}^{M}v_k^{i,n,j}g(x_{i+Rk}),\\\hat{\Upsilon}_{g,M}^{n,j}&=\mathbb{E}_{t_n}^x[g(\bar{X}^{n,j})]-\sum_{k=-M}^{M}\lambda_k^{i,n,j,M}g(x_{i+Rk}).\end{aligned}$$

By removing $\hat{Y}^{n,j}_{g,M}$ in (3.17) and defining

$$\hat{\mathbb{E}}^x_{t_n}[g(\bar{X}^{n,j})] = \sum_{k=-M}^{M} \lambda_k^{i,n,j,M} g(x_{i+Rk}),$$

we easily obtain

$$\begin{aligned}\mathbb{E}^x_{t_n}[g(\bar{X}^{n,j})(\Delta W_{n,j})^\top] &= \hat{\mathbb{E}}^x_{t_n}[g(\bar{X}^{n,j})(\Delta W_{n,j})^\top] + \hat{Y}^{n,j}_{g,\omega,M} \\ &= \sum_{k=-M}^{M} \lambda_k^{i,n,j,M} g(x_{i+Rk})(Q^{i,n,j}_{k,H})^\top + \hat{Y}^{n,j}_{g,\omega,M},\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}&Q^{i,n,j}_{k,H} = (\sigma_{t_n})^{-1}(Hk - b_{t_n}\sqrt{\Delta t_{n,j}}), \\ &\hat{Y}^{n,j}_{g,\omega,M} = \mathbb{E}^x_{t_n}[g(\bar{X}^{n,j})(\Delta W_{n,j})^\top] - \sum_{k=-M}^{M} \lambda_k^{i,n,j,M} g(x_{i+Rk})(Q^{i,n,j}_{k,H})^\top.\end{aligned}$$

Note that under certain conditions, the Sinc quadrature truncation errors in (3.17) and (3.18) have the estimates

$$\max_{\substack{0\le n\le N-k \\ 1\le j\le k}} |\hat{Y}^{n,j}_{g,M}| \le \tilde{C}\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big), \quad \max_{\substack{0\le n\le N-k \\ 1\le j\le k}} |\hat{Y}^{n,j}_{g,\omega,M}| \le \tilde{C}Mh^2\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big), \tag{3.19}$$

where $\tilde{C} > 0$ depending on the upper bounds of f and φ. The further details may refer to [25].

Now let Y^n, Z^n, A^n and Γ^n denote the approximated values of Y_t, Z_t, A_t and Γ_t at the time-space point (t_n,x). By using $\hat{\mathbb{E}}^x_{t_n}[\cdot]$ to approximate $\mathbb{E}^x_{t_n}[\cdot]$ in Scheme 3.1, we obtain the fully discrete explicit multistep Scheme 3.2 for solving decoupled 2FBSDEs (1.1) as follows.

Scheme 3.2. Given Y^{N-j} and Z^{N-j}, $j=0,\cdots,k-1$ on $\mathcal{D}^d_h$, for $n=N-k,\cdots,0$ and each $x\in\mathcal{D}^d_h$, solve $\bar{X}^{n,j}$, $Y^n = Y^n(x)$, $Z^n = Z^n(x)$, $A^n = A^n(x)$ and $\Gamma^n = \Gamma^n(x)$ by

$$\bar{X}^{n,j} = x + b(t_n,x)\Delta t_{n,j} + \sigma(t_n,x)\Delta W_{n,j}, \qquad j=1,\cdots,k, \tag{3.20a}$$

$$Z^n = \sum_{j=1}^{k} \alpha^n_{k,j} \hat{\mathbb{E}}^x_{t_n}[\bar{Y}^{n+j}(\Delta W_{n,j})^\top], \tag{3.20b}$$

$$A^n = \sum_{j=0}^{k} \alpha^n_{k,j} \hat{\mathbb{E}}^x_{t_n}[\bar{Z}^{n+j}], \tag{3.20c}$$

$$\Gamma^n = \sum_{j=1}^{k} \alpha^n_{k,j} \hat{\mathbb{E}}^x_{t_n}[(\bar{Z}^{n+j})^\top(\Delta W_{n,j})^\top], \tag{3.20d}$$

$$\sum_{j=0}^{k} \alpha^n_{k,j} \hat{\mathbb{E}}^x_{t_n}[\bar{Y}^{n+j}] = -f(t_n,x,Y^n_{k-1},Z^n,\Gamma^n), \tag{3.20e}$$

with $\bar{Y}^{n+j}=Y^{n+j}(\bar{X}^{n,j})$, $\bar{Z}^{n+j}=Z^{n+j}(\bar{X}^{n,j})$, and Y_{k-1}^n is recursively calculated by

$$\alpha_{i,0}^n Y_i^n=-\sum_{j=1}^{i}\alpha_{i,j}^n\hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+j}]-f(t_n,x,Y_{i-1}^n,Z^n,\Gamma^n),\quad i=1,2,\cdots,k-1, \tag{3.21}$$

with $Y_0^n=\hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+1}]$.

Remark 3.2. Now for the solution $(X_t,Y_t,Z_t,\Gamma_t,A_t)$ of the decoupled 2FBSDEs (1.1), we have the identities

$$\begin{aligned}
&\sum_{j=0}^{k}\alpha_{k,j}^n\hat{\mathbb{E}}_{t_n}^x[\bar{Y}_{t_{n+j}}]=-f(t_n,X_{t_n},Y_{t_n,k-1},Z_{t_n},\Gamma_{t_n})+R_y^n,\\
&\sum_{j=0}^{k}\alpha_{k,j}^n\hat{\mathbb{E}}_{t_n}^x[\bar{Z}_{t_{n+j}}]=A_{t_n}+R_A^n,\\
&\sum_{j=1}^{k}\alpha_{k,j}^n\hat{\mathbb{E}}_{t_n}^x[\bar{Y}_{t_{n+j}}(\Delta W_{n,j})^\top]=Z_{t_n}^\top+R_z^n,\\
&\sum_{j=1}^{k}\alpha_{k,j}^n\hat{\mathbb{E}}_{t_n}^x[\bar{Z}_{t_{n+j}}^\top(\Delta W_{n,j})^\top]=\Gamma_{t_n}+R_\Gamma^n,
\end{aligned}$$

where R_y^n, R_z^n, R_A^n and R_Γ^n are the local truncation errors of Scheme 3.2 defined as

$$\begin{aligned}
&R_y^n=\bar{R}_{y,k}^n-\sum_{j=0}^{k}\alpha_{k,j}^n\mathrm{Y}_{y,M}^{n,j}, &&R_z^n=R_{z,k}^n-\sum_{j=1}^{k}\alpha_{k,j}^n\hat{\mathrm{Y}}_{y,\omega,M}^{n,j},\\
&R_A^n=R_{A,k}^n-\sum_{j=0}^{k}\alpha_{k,j}^n\mathrm{Y}_{z,M}^{n,j}, &&R_\Gamma^n=R_{\Gamma,k}^n-\sum_{j=1}^{k}\alpha_{k,j}^n\hat{\mathrm{Y}}_{z,\omega,M}^{n,j},
\end{aligned}$$

with $\bar{R}_{y,k}^n$, $R_{A,k}^n$, $R_{z,k}^n$ and $R_{\Gamma,k}^n$ defined in (2.14) and (2.10b)-(2.10d), respectively. Under certain conditions, by (3.3) and (3.19), it holds that

$$R_y^n=\mathcal{O}((\Delta t)^k)+\mathcal{O}\Big(\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big)\Big), \tag{3.22a}$$

$$R_z^n=\mathcal{O}((\Delta t)^k)+\mathcal{O}\Big(Mh^2\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big)\Big), \tag{3.22b}$$

$$R_A^n=\mathcal{O}((\Delta t)^k)+\mathcal{O}\Big(\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big)\Big), \tag{3.22c}$$

$$R_\Gamma^n=\mathcal{O}((\Delta t)^k)+\mathcal{O}\Big(Mh^2\exp\Big(-\frac{M^2h^2}{2k|\sigma|^2}\Big)\Big). \tag{3.22d}$$

3.3 Explicit multistep scheme for coupled 2FBSDEs

Now we extend Scheme 3.2 to solve the fully coupled 2FBSDEs (1.1) with the forward diffusion process X_t as

$$X_t = X_0 + \int_0^t b(s,X_s,Y_s,Z_s,\Gamma_s)\mathrm{d}s + \int_0^t \sigma(s,X_s,Y_s,Z_s,\Gamma_s)\mathrm{d}W_s. \tag{3.23}$$

The explicit Scheme 3.3 for coupled 2FBSDEs can be directly proposed as follows.

Scheme 3.3. Given Y^{N-j} and Z^{N-j}, $j=0,\cdots,k-1$ on $\mathcal{D}_h^d$, for $n=N-k,\cdots,0$ and each $x\in\mathcal{D}_h^d$, solve $Y^n=Y^n(x)$, $Z^n=Z^n(x)$, $A^n=A^n(x)$ and $\Gamma^n=\Gamma^n(x)$ by

$$\bar{X}^{n,j} = x + b(t_n,x,Y^n,Z^n,\Gamma^n)\Delta t_{n,j} + \sigma(t_n,x,Y^n,Z^n,\Gamma^n)\Delta W_{n,j}, \qquad j=1,\cdots,k, \tag{3.24a}$$

$$Z^n = \sum_{j=1}^k \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+j}(\Delta W_{n,j})^\top], \tag{3.24b}$$

$$A^n = \sum_{j=0}^k \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x[\bar{Z}^{n+j}], \tag{3.24c}$$

$$\Gamma^n = \sum_{j=1}^k \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x[(\bar{Z}^{n+j})^\top(\Delta W_{n,j})^\top], \tag{3.24d}$$

$$\sum_{j=0}^k \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+j}] = -f(t_n,x,Y_{k-1}^n,Z^n,\Gamma^n), \tag{3.24e}$$

with $\bar{Y}^{n+j}=Y^{n+j}(\bar{X}^{n,j})$, $\bar{Z}^{n+j}=Z^{n+j}(\bar{X}^{n,j})$ and Y_{k-1}^n is recursively calculated by

$$\alpha_{i,0}^n Y_i^n = -\sum_{j=1}^i \alpha_{i,j}^n \hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+j}] - f(t_n,x,Y_{i-1}^n,Z^n,\Gamma^n), \quad i=1,2,\cdots,k-1, \tag{3.25}$$

with $Y_0^n = \hat{\mathbb{E}}_{t_n}^x[\bar{Y}^{n+1}]$.

We note that Scheme 3.3 becomes Scheme 3.2 when the drift and diffusion coefficients b and σ are independent of Y, Z and Γ. As the dynamic complexity of coupled 2FBSDEs, in Scheme 3.3, some iterative methods are still required in solving $\bar{X}^{n,j}$. Here we propose the iterative scheme for solving $\bar{X}^{n,j}$, Y^n, Z^n, Γ^n and A^n as follows.

Scheme 3.4. Given Y^{N-j}, Z^{N-j} and Γ^{N-j}, $j=0,\cdots,k-1$ on $\mathcal{D}_h^d$, for $n=N-k,\cdots,0$ and each $x\in\mathcal{D}_h^d$, solve $Y^n=Y^n(x)$, $Z^n=Z^n(x)$, $A^n=A^n(x)$ and $\Gamma^n=\Gamma^n(x)$ by

1. Let $Y_{k-1}^{n,0}=Y^{n+1}(x)$, $Z^{n,0}=Z^{n+1}(x)$ and $\Gamma^{n,0}=\Gamma^{n+1}(x)$;

2. For $l=0,1,\cdots$, solve $\bar{X}^{n,j}$ $(j=1,2,\cdots,k)$, $Y_{k-1}^{n,l+1}=Y_{k-1}^{n,l+1}(x)$, $Z^{n,l+1}=Z^{n,l+1}(x)$, $A^{n,l+1}=A^{n,l+1}(x)$ and $\Gamma^{n,l+1}=\Gamma^{n,l+1}(x)$ by

$$\begin{aligned}
&\bar{X}^{n,j}=x+b(t_n,x,Y_{k-1}^{n,l},Z^{n,l},\Gamma^{n,l})\Delta t_{n,j}+\sigma(t_n,x,Y_{k-1}^{n,l},Z^{n,l},\Gamma^{n,l})\Delta W_{n,j},\\
&Z^{n,l+1}=\sum_{j=1}^{k}\alpha_{k,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[\bar{Y}^{n+j}(\Delta W_{n,j})^{\top}],\\
&A^{n,l+1}=\alpha_{k,0}^{n}Z^{n,l+1}+\sum_{j=1}^{k}\alpha_{k,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[\bar{Z}^{n+j}],\\
&\Gamma^{n,l+1}=\sum_{j=1}^{k}\alpha_{k,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[(\bar{Z}^{n+j})^{\top}(\Delta W_{n,j})^{\top}],\\
&\alpha_{i,0}^{n}Y_{i}^{n,l+1}=-\sum_{j=1}^{i}\alpha_{i,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[\bar{Y}^{n+j}]-f(t_n,x,Y_{i-1}^{n,l+1},Z^{n,l+1},\Gamma^{n,l+1}),
\end{aligned}$$

with $\bar{Y}^{n+j}=Y^{n+j}(\bar{X}^{n,j})$, $\bar{Z}^{n+j}=Z^{n+j}(\bar{X}^{n,j})$, and $Y_0^{n,l+1}=\hat{\mathbb{E}}_{t_n}^{x}[\bar{Y}^{n+1}]$ for $i=1,\cdots,k-1$, until

$$\max\Big\{|Y_{k-1}^{n,l+1}-Y_{k-1}^{n,l}|,|Z^{n,l+1}-Z^{n,l}|,|A^{n,l+1}-A^{n,l}|,|\Gamma^{n,l+1}-\Gamma^{n,l}|\Big\}<\varepsilon;$$

3. Let $Y_{k-1}^{n}=Y_{k-1}^{n,l+1}(x)$, $Z^n=Z^{n,l+1}(x)$, $A^n=A^{n,l+1}(x)$, $\Gamma^n=\Gamma^{n,l+1}(x)$, and solve $Y^n=Y^n(x)$ by

$$\alpha_{k,0}^{n}Y^{n}=-\sum_{j=1}^{k}\alpha_{k,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[\bar{Y}^{n+j}]-f(t_n,x,Y_{k-1}^{n},Z^{n},\Gamma^{n}).$$

For comparison, we write the following implicit multistep Scheme 3.5, which was proposed in [33] by directly removing the terms $R_{y,k}^n$, $R_{z,k}^n$, $R_{A,k}^n$ and $R_{\Gamma,k}^n$ in (2.10a)-(2.10d).

Scheme 3.5. Given Y^{N-j}, Z^{N-j} and Γ^{N-j}, $j=0,\cdots,k-1$ on $\mathcal{D}_h^d$, for $n=N-k,\cdots,0$ and each $x\in\mathcal{D}_h^d$, solve $Y^n=Y^n(x)$, $Z^n=Z^n(x)$, $A^n=A^n(x)$ and $\Gamma^n=\Gamma^n(x)$ by

1. Let $Y^{n,0}=Y^{n+1}(x)$, $Z^{n,0}=Z^{n+1}(x)$ and $\Gamma^{n,0}=\Gamma^{n+1}(x)$;
2. For $l=0,1,\cdots$, solve $\bar{X}^{n,j}$ $(j=1,2,\cdots,k)$, $Y^{n,l+1}=Y^{n,l+1}(x)$, $Z^{n,l+1}=Z^{n,l+1}(x)$, $A^{n,l+1}=A^{n,l+1}(x)$ and $\Gamma^{n,l+1}=\Gamma^{n,l+1}(x)$ by

$$\begin{aligned}
&\bar{X}^{n,j}=x+b(t_n,x,Y^{n,l},Z^{n,l},\Gamma^{n,l})\Delta t_{n,j}+\sigma(t_n,x,Y^{n,l},Z^{n,l},\Gamma^{n,l})\Delta W_{n,j},\\
&Z^{n,l+1}=\sum_{j=1}^{k}\alpha_{k,j}^{n}\hat{\mathbb{E}}_{t_n}^{x}[\bar{Y}^{n+j}(\Delta W_{n,j})^{\top}],
\end{aligned}$$

$$A^{n,l+1} = \alpha_{k,0}^n Z^{n,l+1} + \sum_{j=1}^{k} \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x [\bar{Z}^{n+j}],$$

$$\Gamma^{n,l+1} = \sum_{j=1}^{k} \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x [(\bar{Z}^{n+j})^\top (\Delta W_{n,j})^\top],$$

$$\alpha_{k,0}^n Y^{n,l+1} = -\sum_{j=1}^{k} \alpha_{k,j}^n \hat{\mathbb{E}}_{t_n}^x [\bar{Y}^{n+j}] - f(t_n, x, Y^{n,l+1}, Z^{n,l+1}, \Gamma^{n,l+1}),$$

with $\bar{Y}^{n+j} = Y^{n+j}(\bar{X}^{n,j})$ and $\bar{Z}^{n+j} = Z^{n+j}(\bar{X}^{n,j})$, until

$$\max\Big\{ |Y^{n,l+1} - Y^{n,l}|, |Z^{n,l+1} - Z^{n,l}|, |A^{n,l+1} - A^{n,l}|, |\Gamma^{n,l+1} - \Gamma^{n,l}| \Big\} < \varepsilon;$$

3. Let $Y^n = Y^{n,l+1}(x)$, $Z^n = Z^{n,l+1}(x)$, $A^n = A^{n,l+1}(x)$ and $\Gamma^n = \Gamma^{n,l+1}(x)$.

Remark 3.3.

1. Both Schemes 3.4 and 3.5 need the values $\hat{\mathbb{E}}_{t_n}^{X^n}[Y^{n+i}]$, $\hat{\mathbb{E}}_{t_n}^{X^n}[Z^{n+i}]$, $\hat{\mathbb{E}}_{t_n}^{X^n}[Y^{n+i}(\Delta W_{n,i})^\top]$ and $\hat{\mathbb{E}}_{t_n}^{X^n}[(Z^{n+i})^\top(\Delta W_{n,i})^\top]$ $(i = 1, \cdots, k)$. The main difference is that Scheme 3.4 solves Y^n explicitly, however Scheme 3.5 solves Y^n implicitly, requiring some iterative methods for solving the solution. This implicit process may be time-consuming.

2. No spatial interpolations are required in the approximation operator $\hat{\mathbb{E}}_{t_n}^x[\cdot]$ by using the Sinc quadrature rule in this paper, while the authors in [33] used the Gaussian-Hermite quadrature rule, which requires spatial interpolations.

4 Numerical tests

In this section, we will carry out some numerical tests to demonstrate the strong stability, high accuracy and high efficiency of our explicit multistep Schemes 3.2 and 3.4 for solving 2FBSDEs. For simplicity, we will use uniform partitions in time. The time interval $[0,T]$ will be uniformly divided into N parts with time step $\Delta t = \frac{T}{N}$, the time grids $t_n = n\Delta t$, $n = 0, 1, \cdots, N$. For all the tests, the terminal time T is set to be 1.0.

The conditional mathematical expectations $\mathbb{E}_{t_n}^{X^n}[\cdot]$ in all schemes are needed to approximate, thus we use the Sinc quadrature rule to approximate the expectations in our tests. The parameters M and h of the Sinc quadrature rule are selected such that $Mh = 8$. In this case, the errors from the rule are negligible (compared with the time discretization errors). For more details about the Sinc quadrature rule for $\mathbb{E}_{t_n}^{X^n}[\cdot]$, readers may refer to [25]. In the sequel, we list the errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, $|A_0 - A^0|$ and $|\Gamma_0 - \Gamma^0|$, the convergence rate (CR) and the running time (RT), respectively. The unit of RT is second. All numerical tests are provided in Python 3.8.19 on a laptop with an Intel Core i9-13900HX 32-Core Processor (2.20GHz) and 16.0 GB of DDR5 RAM (5600MT).

Example 4.1. In this test, we will test the accuracy and efficiency of Scheme 3.2 for solving the decoupled FBSDEs. The decoupled 2FBSDEs are

$$\begin{cases} \mathrm{d}X_t = \sin(t+X_t)\mathrm{d}t + 0.2\cos(t+X_t)\mathrm{d}W_t, \\ -\mathrm{d}Y_t = \big(-5\cos(t+X_t)Z_t - \cos(t+X_t)(Y_t^2+Y_t) - 0.25\Gamma_t\big)\mathrm{d}t - Z_t\mathrm{d}W_t, \\ \mathrm{d}Z_t = A_t\mathrm{d}t + \Gamma_t\mathrm{d}W_t, \end{cases} \tag{4.1}$$

with the initial and terminal conditions $X_0 = 0.25$ and $Y_T = \sin(T+X_T)$, respectively. The analytic solutions Y_t, Z_t, A_t and Γ_t are

$$\begin{aligned} &Y_t = \sin(t+X_t), \quad Z_t = 0.2\cos^2(t+X_t), \\ &A_t = -0.2\sin(2t+2X_t)(1+\sin(t+X_t)) - 0.008\cos(2t+2X_t)\cos^2(t+X_t), \\ &\Gamma_t = -0.08\sin(t+X_t)\cos^2(t+X_t). \end{aligned}$$

We solve the numerical solutions Y^0, Z^0, A^0 and Γ^0 by Schemes 3.2 and 3.5 for $k = 1,2,\cdots,6$. Their errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, $|A_0 - A^0|$, $|\Gamma_0 - \Gamma^0|$, CR and RT are listed in Tables 2 and 3, respectively. The numerical results in the tables show that:

1. The explicit multistep Scheme 3.2 is stable, effective for solving the decouple 2FBSDEs with order-k for $1 \le k \le 6$.
2. Compared with the implicit Scheme 3.5, the explicit Scheme 3.2 is more efficient for the same step k, achieving time savings at least 11 times, and the running times increase monotonically with respect to N and k.

Example 4.2. We now test the accuracy and efficiency of explicit multistep Scheme 3.4 for solving fully coupled 2FBSDEs. The coupled 2FBSDEs are

$$\begin{cases} \mathrm{d}X_t = (\sin(t+X_t) + 5Z_t + \sin(t+X_t)Y_t - 1)\mathrm{d}t \\ \qquad\qquad + (0.2\cos(t+X_t) - 0.2 + 0.2\cos^2(t+X_t) + 0.2\sin(t+X_t)Y_t)\mathrm{d}W_t, \\ \mathrm{d}Y_t = -5\cos(t+X_t)Z_t - \cos(t+X_t)(Y_t^2+Y_t) - 0.25\Gamma_t\mathrm{d}t - Z_t\mathrm{d}W_t, \\ \mathrm{d}Z_t = A_t\mathrm{d}t + \Gamma_t\mathrm{d}W_t, \end{cases}$$

with $X_0 = 0.25$, $Y_T = \sin(T+X_T)$. The analytic solution Y_t, Z_t, A_t and Γ_t are

$$\begin{aligned} &Y_t = \sin(t+X_t), \quad Z_t = 0.2\cos^2(t+X_t), \\ &A_t = -0.2\sin(2t+2X_t)(1+\sin(t+X_t)) - 0.008\cos(2t+2X_t)\cos^2(t+X_t), \\ &\Gamma_t = -0.08\sin(t+X_t)\cos^2(t+X_t). \end{aligned}$$

The errors, CR, and RT by Schemes 3.4 and 3.5 are listed in Tables 4 and 5. By the results in these two tables, we have the conclusions as follows.

1. The results in Table 4 clearly show that the explicit multistep Scheme 3.4 is an order-k scheme with $1 \le k \le 6$ for solving the coupled 2FBSDEs.

2. By comparing Tables 4 and 5, for the same k, the explicit multistep Scheme 3.4 is more efficient than the implicit Scheme 3.5, and can save at least 12 times. The running times depend on both the step number k and the time partition number N, and increase with respect to N and k.

Example 4.3. This example will show the capacity of the explicit multistep scheme for solving the stochastic optimal control problems (SOCPs) by means of 2FBSDEs. Consider

Table 2: Numerical results for Example 4.1 obtained by the explicit multistep Scheme 3.2.

		$N=30$	$N=40$	$N=50$	$N=60$	$N=70$	CR
$k=1$	$\|Y_0-Y^0\|$	2.790E-02	2.136E-02	1.733E-02	1.457E-02	1.255E-02	0.942
	$\|Z_0-Z^0\|$	8.799E-03	6.766E-03	5.466E-03	4.541E-03	3.895E-03	0.963
	$\|A_0-A^0\|$	6.342E-02	4.521E-02	3.519E-02	2.983E-02	2.627E-02	1.046
	$\|\Gamma_0-\Gamma^0\|$	4.756E-03	3.522E-03	2.793E-03	2.337E-03	2.009E-03	1.018
	RT	0.010s	0.011s	0.022s	0.034s	0.057s	
$k=2$	$\|Y_0-Y^0\|$	6.462E-04	3.841E-04	2.527E-04	1.768E-04	1.309E-04	1.886
	$\|Z_0-Z^0\|$	6.066E-04	3.482E-04	2.289E-04	1.640E-04	1.241E-04	1.873
	$\|A_0-A^0\|$	6.745E-03	4.659E-03	1.877E-03	1.746E-03	1.555E-03	1.897
	$\|\Gamma_0-\Gamma^0\|$	4.854E-04	2.497E-04	1.533E-04	1.146E-04	8.864E-05	2.010
	RT	0.018s	0.028s	0.040s	0.053s	0.084s	
$k=3$	$\|Y_0-Y^0\|$	3.432E-05	1.497E-05	8.407E-06	5.015E-06	3.273E-06	2.760
	$\|Z_0-Z^0\|$	2.147E-05	8.572E-06	4.352E-06	2.598E-06	1.656E-06	3.018
	$\|A_0-A^0\|$	5.121E-04	2.333E-04	1.378E-04	7.856E-05	5.689E-05	2.693
	$\|\Gamma_0-\Gamma^0\|$	5.683E-05	2.693E-05	1.362E-05	8.300E-06	5.333E-06	2.810
	RT	0.021s	0.030s	0.046s	0.071s	0.095s	
$k=4$	$\|Y_0-Y^0\|$	9.914E-07	3.345E-07	1.520E-07	7.617E-08	4.270E-08	3.699
	$\|Z_0-Z^0\|$	5.118E-07	1.431E-07	6.758E-08	3.311E-08	1.804E-08	3.894
	$\|A_0-A^0\|$	4.444E-06	1.505E-06	7.037E-07	3.525E-07	1.957E-07	3.662
	$\|\Gamma_0-\Gamma^0\|$	2.265E-06	8.760E-07	3.629E-07	2.162E-07	8.051E-08	3.803
	RT	0.025s	0.042s	0.062s	0.083s	0.116s	
$k=5$	$\|Y_0-Y^0\|$	1.333E-08	4.211E-09	1.400E-09	6.298E-10	2.967E-10	4.513
	$\|Z_0-Z^0\|$	1.606E-08	4.429E-09	1.585E-09	6.283E-10	3.043E-10	4.695
	$\|A_0-A^0\|$	2.090E-06	6.152E-07	1.444E-07	1.129E-07	3.837E-08	4.620
	$\|\Gamma_0-\Gamma^0\|$	1.654E-07	4.808E-08	2.130E-08	8.837E-09	2.302E-09	4.796
	RT	0.049s	0.095s	0.160s	0.240s	0.340s	
$k=6$	$\|Y_0-Y^0\|$	8.794E-10	1.693E-10	5.327E-11	2.175E-11	6.306E-12	5.652
	$\|Z_0-Z^0\|$	1.335E-10	2.630E-11	8.826E-12	3.798E-12	9.200E-13	5.616
	$\|A_0-A^0\|$	8.978E-07	1.878E-07	6.679E-08	2.226E-08	7.517E-09	5.533
	$\|\Gamma_0-\Gamma^0\|$	7.602E-09	1.437E-09	4.761E-10	1.900E-10	6.032E-11	5.550
	RT	0.076s	0.147s	0.249s	0.370s	0.514s	

Table 3: Numerical results for Example 4.1 obtained by the implicit multistep Scheme 3.5.

		$N=30$	$N=40$	$N=50$	$N=60$	$N=70$	CR
$k=1$	$\|Y_0-Y^0\|$	4.331E-03	3.075E-03	2.340E-03	1.897E-03	1.603E-03	1.179
	$\|Z_0-Z^0\|$	8.901E-03	6.464E-03	5.095E-03	4.243E-03	3.625E-03	1.058
	$\|A_0-A^0\|$	6.712E-02	6.104E-02	4.953E-02	3.982E-02	2.845E-02	0.981
	$\|\Gamma_0-\Gamma^0\|$	7.039E-03	5.518E-03	4.522E-03	3.807E-03	3.286E-03	0.899
	RT	0.059s	0.078s	0.186s	0.355s	0.730s	
$k=2$	$\|Y_0-Y^0\|$	2.323E-04	1.353E-04	8.806E-05	6.257E-05	4.655E-05	1.898
	$\|Z_0-Z^0\|$	4.506E-04	2.575E-04	1.673E-04	1.191E-04	8.924E-05	1.912
	$\|A_0-A^0\|$	5.591E-03	2.684E-03	1.844E-03	1.313E-03	9.961E-04	2.002
	$\|\Gamma_0-\Gamma^0\|$	5.846E-05	3.327E-05	1.788E-05	1.464E-05	1.028E-05	2.057
	RT	0.109s	0.207s	0.366s	0.601s	1.165s	
$k=3$	$\|Y_0-Y^0\|$	3.052E-06	1.160E-06	7.541E-07	4.456E-07	3.095E-07	2.652
	$\|Z_0-Z^0\|$	1.051E-05	4.340E-06	2.218E-06	1.315E-06	8.526E-07	2.968
	$\|A_0-A^0\|$	2.899E-04	2.115E-04	1.143E-04	5.313E-05	2.777E-05	2.804
	$\|\Gamma_0-\Gamma^0\|$	3.393E-05	1.575E-05	8.229E-06	5.009E-06	3.194E-06	2.792
	RT	0.131s	0.223s	0.435s	0.841s	1.365s	
$k=4$	$\|Y_0-Y^0\|$	4.288E-07	1.419E-07	6.478E-08	3.285E-08	1.831E-08	3.701
	$\|Z_0-Z^0\|$	5.643E-07	1.840E-07	8.434E-08	4.329E-08	2.415E-08	3.694
	$\|A_0-A^0\|$	7.348E-06	2.818E-06	1.482E-06	7.006E-07	3.133E-07	3.624
	$\|\Gamma_0-\Gamma^0\|$	1.323E-06	4.872E-07	2.034E-07	1.313E-07	4.960E-08	3.723
	RT	0.169s	0.332s	0.606s	1.013s	1.751s	
$k=5$	$\|Y_0-Y^0\|$	1.934E-08	5.442E-09	1.864E-09	8.005E-10	3.921E-10	4.620
	$\|Z_0-Z^0\|$	1.011E-08	2.815E-09	1.006E-09	4.105E-10	2.098E-10	4.602
	$\|A_0-A^0\|$	1.301E-06	4.108E-07	1.498E-07	6.065E-08	2.638E-08	4.589
	$\|\Gamma_0-\Gamma^0\|$	9.957E-08	3.205E-08	1.321E-08	5.409E-09	1.640E-09	4.683
	RT	0.455s	1.055s	2.088s	3.722s	6.145s	
$k=6$	$\|Y_0-Y^0\|$	1.747E-10	3.297E-11	1.026E-11	4.007E-12	1.275E-12	5.670
	$\|Z_0-Z^0\|$	5.899E-10	1.185E-10	3.972E-11	1.555E-11	3.751E-12	5.728
	$\|A_0-A^0\|$	3.266E-07	6.045E-08	2.309E-08	9.299E-09	2.384E-09	5.535
	$\|\Gamma_0-\Gamma^0\|$	7.602E-09	1.437E-09	4.761E-10	1.900E-10	6.032E-11	5.550
	RT	0.762s	1.742s	3.355s	5.941s	9.719s	

a controlled diffusion process X_t, governed by the SDE

$$dX_t = b\alpha_t dt + \sigma dW_t, \tag{4.2}$$

where X_t is the controlled state process representing the distance between the particle and the focus of the microscope, α_t is the control variable, $b \in \mathbb{R}$ denotes the gain in our servo loop and $\sigma > 0$ is the diffusion constant of the particle. The admissible control set $\mathcal{U}$ for the α_t is defined as

$$\mathcal{U} = \{\alpha_\cdot \in \mathcal{M}^2(\mathbb{R}) | \alpha_t \in U,\ t \in [0,T]\},$$

Table 4: Numerical results for Example 4.2 obtained by the explicit multistep Scheme 3.4.

		$N=20$	$N=30$	$N=40$	$N=50$	$N=60$	CR
$k=1$	$\lvert Y_0-Y^0\rvert$	6.944E-02	4.835E-02	3.700E-02	3.007E-02	2.532E-02	0.919
	$\lvert Z_0-Z^0\rvert$	1.391E-02	9.931E-03	7.625E-03	6.323E-03	5.301E-03	0.877
	$\lvert A_0-A^0\rvert$	4.590E-02	3.346E-02	2.528E-02	1.859E-02	1.657E-02	0.961
	$\lvert \Gamma_0-\Gamma^0\rvert$	6.916E-03	4.757E-03	3.675E-03	2.961E-03	2.463E-03	0.935
	RT	0.019s	0.035s	0.056s	0.085s	0.110s	
$k=2$	$\lvert Y_0-Y^0\rvert$	2.877E-03	1.334E-03	7.640E-04	4.940E-04	3.462E-04	1.929
	$\lvert Z_0-Z^0\rvert$	2.409E-04	9.282E-05	4.696E-05	2.878E-05	1.804E-05	2.345
	$\lvert A_0-A^0\rvert$	5.736E-03	2.861E-03	1.587E-03	1.092E-03	5.151E-04	2.091
	$\lvert \Gamma_0-\Gamma^0\rvert$	9.481E-04	4.011E-04	2.200E-04	1.475E-04	9.649E-05	2.059
	RT	0.053s	0.106s	0.172s	0.291s	0.434s	
$k=3$	$\lvert Y_0-Y^0\rvert$	1.561E-05	5.549E-06	2.090E-06	9.435E-07	5.373E-07	3.115
	$\lvert Z_0-Z^0\rvert$	5.826E-05	1.863E-05	7.988E-06	4.118E-06	2.393E-06	2.909
	$\lvert A_0-A^0\rvert$	5.474E-04	1.489E-04	9.484E-05	4.629E-05	2.376E-05	2.736
	$\lvert \Gamma_0-\Gamma^0\rvert$	7.390E-05	1.666E-05	7.326E-06	4.532E-06	2.677E-06	2.974
	RT	0.090s	0.205s	0.371s	0.604s	0.793s	
$k=4$	$\lvert Y_0-Y^0\rvert$	8.391E-07	1.874E-07	6.375E-08	2.771E-08	1.377E-08	3.738
	$\lvert Z_0-Z^0\rvert$	1.601E-06	3.575E-07	1.192E-07	5.022E-08	2.452E-08	3.804
	$\lvert A_0-A^0\rvert$	5.064E-05	1.426E-05	3.631E-06	1.569E-06	6.673E-07	3.974
	$\lvert \Gamma_0-\Gamma^0\rvert$	6.442E-06	9.683E-07	3.200E-07	1.853E-07	7.679E-08	3.919
	RT	0.140s	0.327s	0.645s	0.950s	1.410s	
$k=5$	$\lvert Y_0-Y^0\rvert$	3.122E-07	5.673E-08	1.576E-08	5.676E-09	1.825E-09	4.602
	$\lvert Z_0-Z^0\rvert$	1.825E-07	3.090E-08	7.921E-09	2.843E-09	1.070E-09	4.654
	$\lvert A_0-A^0\rvert$	7.796E-07	1.229E-07	3.054E-08	1.241E-08	3.887E-09	4.737
	$\lvert \Gamma_0-\Gamma^0\rvert$	3.850E-07	6.376E-08	1.698E-08	6.019E-09	2.482E-09	4.585
	RT	0.194s	0.488s	0.936s	1.372s	2.054s	
$k=6$	$\lvert Y_0-Y^0\rvert$	1.804E-08	4.603E-09	5.339E-10	1.506E-10	4.793E-11	5.552
	$\lvert Z_0-Z^0\rvert$	1.701E-08	5.696E-09	6.335E-10	1.708E-10	4.219E-11	5.581
	$\lvert A_0-A^0\rvert$	5.143E-08	1.625E-08	1.966E-09	5.419E-10	1.213E-10	5.575
	$\lvert \Gamma_0-\Gamma^0\rvert$	4.017E-08	1.327E-08	1.389E-09	4.295E-10	9.948E-11	5.560
	RT	0.258s	0.687s	1.213s	1.961s	2.959s	

where $\mathcal{M}^2(\mathbb{R})$ denotes the space of all the $\mathcal{F}_t$-adapted process valued in $\mathbb{R}$:

$$\mathbb{E}\Big[\int_0^T |X_t|^2 \mathrm{d}t\Big] < \infty.$$

Here U is a nonempty, convex and closed subset of $\mathbb{R}$. For $\alpha_\cdot \in \mathcal{U}$, it is generally known that (4.2) admits a unique solution X_t under certain conditions on b and σ. Our goal is to keep the particle in focus, that is, we aim to maintain it as close to the origin as possible. We need to impose a power constraint on the control, since the servo motor

Table 5: Numerical results for Example 4.2 obtained by the implicit multistep Scheme 3.5.

		$N=20$	$N=30$	$N=40$	$N=50$	$N=60$	CR
$k=1$	$\lvert Y_0-Y^0\rvert$	2.934E-02	2.065E-02	1.592E-02	1.296E-02	1.092E-02	0.900
	$\lvert Z_0-Z^0\rvert$	1.585E-02	1.081E-02	8.178E-03	6.588E-03	5.511E-03	0.962
	$\lvert A_0-A^0\rvert$	2.580E-02	1.897E-02	1.444E-02	1.163E-02	8.911E-03	0.948
	$\lvert \Gamma_0-\Gamma^0\rvert$	6.974E-03	4.801E-03	3.659E-03	2.952E-03	2.475E-03	0.943
	RT	0.137s	0.292s	0.556s	1.019s	1.557s	
$k=2$	$\lvert Y_0-Y^0\rvert$	8.576E-04	4.065E-04	2.358E-04	1.536E-04	1.081E-04	1.886
	$\lvert Z_0-Z^0\rvert$	5.004E-04	2.274E-04	1.294E-04	8.400E-05	5.808E-05	1.957
	$\lvert A_0-A^0\rvert$	3.332E-03	1.572E-03	9.090E-04	5.381E-04	3.609E-04	2.021
	$\lvert \Gamma_0-\Gamma^0\rvert$	4.384E-04	1.869E-04	1.031E-04	6.633E-05	4.516E-05	2.063
	RT	0.393s	0.917s	1.830s	3.739s	6.599s	
$k=3$	$\lvert Y_0-Y^0\rvert$	9.797E-05	3.305E-05	1.501E-05	8.025E-06	4.773E-06	2.749
	$\lvert Z_0-Z^0\rvert$	6.031E-05	1.943E-05	8.513E-06	4.457E-06	2.621E-06	2.855
	$\lvert A_0-A^0\rvert$	1.730E-04	5.668E-05	2.695E-05	1.258E-05	8.097E-06	2.810
	$\lvert \Gamma_0-\Gamma^0\rvert$	5.655E-05	1.754E-05	8.110E-06	4.391E-06	2.583E-06	2.792
	RT	0.691s	1.822s	4.102s	8.084s	12.597s	
$k=4$	$\lvert Y_0-Y^0\rvert$	2.715E-06	6.803E-07	2.407E-07	1.055E-07	5.323E-08	3.581
	$\lvert Z_0-Z^0\rvert$	1.393E-06	3.346E-07	1.171E-07	5.120E-08	2.586E-08	3.630
	$\lvert A_0-A^0\rvert$	3.683E-05	8.355E-06	2.489E-06	1.173E-06	5.954E-07	3.777
	$\lvert \Gamma_0-\Gamma^0\rvert$	4.455E-06	1.001E-06	2.889E-07	1.158E-07	4.924E-08	4.098
	RT	1.168s	3.062s	7.398s	12.998s	22.912s	
$k=5$	$\lvert Y_0-Y^0\rvert$	3.265E-07	5.309E-08	1.511E-08	5.398E-09	2.034E-09	4.575
	$\lvert Z_0-Z^0\rvert$	2.149E-07	3.503E-08	9.431E-09	3.243E-09	1.354E-09	4.609
	$\lvert A_0-A^0\rvert$	3.985E-06	6.912E-07	1.862E-07	6.544E-08	2.615E-08	4.562
	$\lvert \Gamma_0-\Gamma^0\rvert$	3.28E-07	6.289E-08	1.604E-08	6.173E-09	2.249E-09	4.590
	RT	2.070s	6.102s	13.604s	23.205s	39.852s	
$k=6$	$\lvert Y_0-Y^0\rvert$	1.625E-08	4.540E-09	5.080E-10	1.504E-10	4.268E-11	5.540
	$\lvert Z_0-Z^0\rvert$	1.431E-08	4.054E-09	4.492E-10	1.321E-10	3.601E-11	5.572
	$\lvert A_0-A^0\rvert$	2.413E-07	6.721E-08	7.819E-09	2.337E-09	6.087E-10	5.544
	$\lvert \Gamma_0-\Gamma^0\rvert$	3.413E-08	9.726E-09	1.067E-09	3.155E-10	8.831E-11	5.556
	RT	3.046s	9.407s	18.956s	35.148s	59.989s	

cannot operate with unlimited input power. The cost functional $J(\alpha_\cdot):\mathcal{U}\to\mathbb{R}$ is

$$J(\alpha_\cdot)=\mathbb{E}\Big[p\int_0^T X_t^2\mathrm{d}t+q\int_0^T \alpha_t^2\mathrm{d}t\Big], \tag{4.3}$$

where p, $q>0$. The SOCP is to find $\alpha_\cdot^*\in\mathcal{U}$ such that $J(\alpha_\cdot^*)$ attains its minimum over the admissible control set, that is,

$$J(\alpha_\cdot^*)=\min_{\alpha_\cdot\in\mathcal{U}}J(\alpha_\cdot),\quad \alpha_\cdot^*=\arg\min_{\alpha_\cdot\in\mathcal{U}}J(\alpha_\cdot),$$

where $\alpha_\cdot^*$ is the optimal control and $J(\alpha_\cdot^*)$ is the optimal value. Let $v(t,x)$ denote the value function as

$$v(t,x)=\min_{\alpha.\in\mathcal{U}} J(t,x,\alpha),$$

where

$$J(t,x,\alpha)=\mathbb{E}\Big[p\int_t^T X_s^2 \mathrm{d}s+q\int_t^T \alpha_s^2 \mathrm{d}s\Big|\mathcal{F}_t,\ X_t=x\Big].$$

By the Bellman's dynamic programming principle [10], the optimal control solution is

$$\alpha_t^*=-\frac{b}{2q}\frac{\partial v(t,x)}{\partial x},$$

where $v(t,x)$ satisfies the fully nonlinear HJB equation

$$\begin{aligned}0=&\frac{\partial v(t,x)}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 v(t,x)}{\partial x^2}+b\alpha_t^*\frac{\partial v(t,x)}{\partial x}+px^2+q(\alpha_t^*)^2\\=&\frac{\partial v(t,x)}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 v(t,x)}{\partial x^2}-\frac{b^2}{4q}\Big(\frac{\partial v(t,x)}{\partial x}\Big)^2+px^2\end{aligned}\tag{4.4}$$

with the terminal condition $v(T,x)=0$.

Now let $(X_t,Y_t,Z_t,\Gamma_t,A_t)$ be the solution of the following coupled 2FBSDEs:

$$\begin{cases}\mathrm{d}X_t=-\dfrac{b^2}{2qc\sigma}Z_t\mathrm{d}t+c\sigma\mathrm{d}W_t,\\-\mathrm{d}Y_t=\Big(\dfrac{1-c^2}{2c^2}\Gamma_t+\dfrac{b^2}{4qc^2\sigma^2}Z_t^2+pX_t^2\Big)\mathrm{d}t-Z_t\mathrm{d}W_t,\\\mathrm{d}Z_t=A_t\mathrm{d}t+\Gamma_t\mathrm{d}W_t,\end{cases}\tag{4.5}$$

where c is a constant parameter. By the nonlinear Feynman-Kac formula of 2FBSDEs [19, 20, 26], it holds that

$$v(t,X_t)=Y_t\quad\text{and}\quad\alpha_t^*=-\frac{b}{2qc\sigma}Z_t,\quad t\in[0,T].$$

Let the initial condition $X_0=0.25$, $b=\sigma=p=0.5$ and $q=1.5$. The errors $|Y_0-Y^0|$, $|Z_0-Z^0|$, $|\Gamma_0-\Gamma^0|$, $|\alpha_0^*-\alpha^0|$ and CR for different values of c by Scheme 3.4 are presented in Tables 6 and 7. Numerical results in Tables 6 and 7 show that the explicit Scheme 3.4 is strongly stable, highly effective and highly accurate for solving the 2FBSDEs and the optimal control α_0^* with k-th order ($1\le k\le 6$).

All the above numerical tests show that the new explicit multistep schemes are stable, efficient and accurate for solving decoupled 2FBSDEs, coupled 2FBSDEs and stochastic optimal control problems.

Table 6: Numerical results for Example 4.3 obtained by the explicit multistep Scheme 3.4 under $c=0.9$.

		$N=30$	$N=40$	$N=50$	$N=60$	$N=70$	CR
$k=1$	$\lvert Y_0-Y^0\rvert$	2.700E-03	2.029E-03	1.625E-03	1.355E-03	1.162E-03	0.995
	$\lvert Z_0-Z^0\rvert$	3.603E-03	2.702E-03	2.162E-03	1.801E-03	1.544E-03	1.000
	$\lvert\Gamma_0-\Gamma^0\rvert$	1.274E-02	9.546E-03	7.633E-03	6.359E-03	5.450E-03	1.002
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.335E-03	1.001E-03	8.006E-04	6.672E-04	5.718E-04	1.000
$k=2$	$\lvert Y_0-Y^0\rvert$	1.545E-06	9.049E-07	5.937E-07	4.193E-07	3.117E-07	1.889
	$\lvert Z_0-Z^0\rvert$	6.282E-06	3.550E-06	2.279E-06	1.585E-06	1.166E-06	1.987
	$\lvert\Gamma_0-\Gamma^0\rvert$	1.936E-05	1.125E-05	7.348E-06	5.171E-06	3.835E-06	1.911
	$\lvert\alpha_0^*-\alpha^0\rvert$	2.327E-06	1.315E-06	8.439E-07	5.871E-07	4.319E-07	1.987
$k=3$	$\lvert Y_0-Y^0\rvert$	5.033E-07	2.270E-07	1.210E-07	7.198E-08	4.622E-08	2.818
	$\lvert Z_0-Z^0\rvert$	5.317E-07	2.347E-07	1.235E-07	7.279E-08	4.645E-08	2.877
	$\lvert\Gamma_0-\Gamma^0\rvert$	1.706E-06	7.511E-07	3.946E-07	2.324E-07	1.482E-07	2.884
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.969E-07	8.691E-08	4.574E-08	2.696E-08	1.720E-08	2.877
$k=4$	$\lvert Y_0-Y^0\rvert$	2.728E-09	9.715E-10	4.277E-10	2.165E-10	1.210E-10	3.677
	$\lvert Z_0-Z^0\rvert$	2.484E-09	8.864E-10	3.906E-10	1.979E-10	1.107E-10	3.672
	$\lvert\Gamma_0-\Gamma^0\rvert$	1.895E-08	6.661E-09	2.909E-09	1.465E-09	8.158E-10	3.712
	$\lvert\alpha_0^*-\alpha^0\rvert$	9.200E-10	3.283E-10	1.447E-10	7.330E-11	4.099E-11	3.672
$k=5$	$\lvert Y_0-Y^0\rvert$	2.278E-10	6.480E-11	2.373E-11	1.029E-11	4.523E-12	4.591
	$\lvert Z_0-Z^0\rvert$	2.231E-10	6.078E-11	2.166E-11	9.278E-12	4.462E-12	4.615
	$\lvert\Gamma_0-\Gamma^0\rvert$	6.427E-10	1.747E-10	6.047E-11	2.874E-11	1.169E-11	4.661
	$\lvert\alpha_0^*-\alpha^0\rvert$	8.262E-11	2.251E-11	8.022E-12	3.436E-12	1.653E-12	4.615
$k=6$	$\lvert Y_0-Y^0\rvert$	2.412E-12	5.503E-13	1.704E-13	6.357E-14	2.044E-14	5.532
	$\lvert Z_0-Z^0\rvert$	3.899E-12	8.777E-13	2.668E-13	9.805E-14	3.219E-14	5.575
	$\lvert\Gamma_0-\Gamma^0\rvert$	2.141E-11	4.773E-12	1.430E-12	5.545E-13	1.041E-13	6.007
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.300E-12	2.925E-13	8.893E-14	3.268E-14	1.073E-14	5.575

5 Conclusions

In this paper, we proposed new explicit multistep schemes for solving 2FBSDEs by introducing a new family of recursively defined processes. In the proposed schemes, we only need calculating the values $\mathbb{E}_{t_n}^{X^n}[Y^{n+i}]$, $\mathbb{E}_{t_n}^{X^n}[Z^{n+i}]$, $\mathbb{E}_{t_n}^{X^n}[Y^{n+i}(\Delta W_{n,i})^\top]$ and $\mathbb{E}_{t_n}^{X^n}[Z^{n+i}(\Delta W_{n,i})^\top]$ $(i=1,\cdots,k)$ to explicitly calculate the generator f, and to solve the approximation solutions (Y^n,Z^n,Γ^n,A^n) explicitly and accurately. The above features extremely improve the efficiency. Our numerical tests are showed that the explicit multistep schemes are strongly stable, highly efficient and highly accurate for solving the 2FBSDEs and stochastic optimal control problems. We believe that the new explicit schemes may have the capacity of efficiently and accurately solving complex 2FBSDEs and their associated problems, such as PDEs, mathematical finance, artificial intelligence, deep learning, and so on.

Table 7: Numerical results for Example 4.3 obtained by the explicit multistep Scheme 3.4 under $c=1.5$.

		$N=30$	$N=40$	$N=50$	$N=60$	$N=70$	CR
$k=1$	$\lvert Y_0-Y^0\rvert$	6.362E-04	4.678E-04	3.698E-04	3.057E-04	2.605E-04	1.054
	$\lvert Z_0-Z^0\rvert$	4.604E-03	3.453E-03	2.762E-03	2.302E-03	1.973E-03	1.000
	$\lvert\Gamma_0-\Gamma^0\rvert$	2.079E-02	1.558E-02	1.246E-02	1.038E-02	8.898E-03	1.002
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.023E-03	7.673E-04	6.138E-04	5.115E-04	4.384E-04	1.000
$k=2$	$\lvert Y_0-Y^0\rvert$	1.873E-06	1.049E-06	6.693E-07	4.637E-07	3.401E-07	2.014
	$\lvert Z_0-Z^0\rvert$	8.027E-06	4.537E-06	2.912E-06	2.026E-06	1.490E-06	1.987
	$\lvert\Gamma_0-\Gamma^0\rvert$	3.161E-05	1.838E-05	1.200E-05	8.442E-06	6.261E-06	1.911
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.784E-06	1.008E-06	6.470E-07	4.501E-07	3.311E-07	1.987
$k=3$	$\lvert Y_0-Y^0\rvert$	2.095E-07	9.459E-08	5.047E-08	3.002E-08	1.929E-08	2.815
	$\lvert Z_0-Z^0\rvert$	6.793E-07	2.998E-07	1.578E-07	9.302E-08	5.936E-08	2.877
	$\lvert\Gamma_0-\Gamma^0\rvert$	2.786E-06	1.226E-06	6.443E-07	3.794E-07	2.419E-07	2.884
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.510E-07	6.663E-08	3.507E-08	2.067E-08	1.319E-08	2.877
$k=4$	$\lvert Y_0-Y^0\rvert$	1.620E-09	5.631E-10	2.441E-10	1.222E-10	6.779E-11	3.746
	$\lvert Z_0-Z^0\rvert$	3.174E-09	1.133E-09	4.992E-10	2.529E-10	1.414E-10	3.672
	$\lvert\Gamma_0-\Gamma^0\rvert$	3.093E-08	1.087E-08	4.750E-09	2.392E-09	1.332E-09	3.712
	$\lvert\alpha_0^*-\alpha^0\rvert$	7.053E-10	2.517E-10	1.109E-10	5.619E-11	3.143E-11	3.672
$k=5$	$\lvert Y_0-Y^0\rvert$	1.019E-10	2.905E-11	1.065E-11	4.599E-12	2.010E-12	4.597
	$\lvert Z_0-Z^0\rvert$	2.850E-10	7.764E-11	2.732E-11	9.952E-12	3.304E-12	5.167
	$\lvert\Gamma_0-\Gamma^0\rvert$	1.050E-09	2.816E-10	9.205E-11	3.630E-11	2.060E-11	4.728
	$\lvert\alpha_0^*-\alpha^0\rvert$	6.334E-11	1.725E-11	6.072E-12	2.211E-12	7.343E-13	5.167
$k=6$	$\lvert Y_0-Y^0\rvert$	1.608E-12	3.954E-13	1.275E-13	4.032E-14	1.388E-14	5.560
	$\lvert Z_0-Z^0\rvert$	5.990E-12	1.364E-12	4.041E-13	2.050E-13	4.191E-14	5.566
	$\lvert\Gamma_0-\Gamma^0\rvert$	8.574E-12	2.033E-12	6.842E-13	2.618E-13	4.777E-14	5.817
	$\lvert\alpha_0^*-\alpha^0\rvert$	1.331E-12	3.032E-13	8.981E-14	4.556E-14	9.313E-15	5.566

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