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Convergence Analysis of a Weak Galerkin Finite Element Method on a Bakhvalov-Type Mesh for a Singularly Perturbed Convection-Diffusion Equation in 2D

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Abstract. In this paper, we propose a weak Galerkin finite element method (WG) for solving singularly perturbed convection-diffusion problems on a Bakhvalov-type mesh in 2D. Our method is flexible and allows the use of discontinuous approximation functions on the mesh. An error estimate is developed in a suitable norm, and the optimal convergence order is obtained. Finally, numerical experiments are conducted to support the theory and to demonstrate the efficiency of the proposed method.

AMS subject classifications: 65N15, 65N30, 35B25

Key words: Weak Galerkin finite element method, convection-diffusion, singularly perturbed, Bakhvalov-type mesh.

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1 Introduction

Consider the following singularly perturbed convection-diffusion problem

$$-\varepsilon\Delta u-\mathbf{b}\cdot\nabla u+cu=f \qquad \text{in } \Omega, \tag{1.1a}$$
$$u=0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

with a positive parameter ε satisfying $0<\varepsilon\ll 1$, and $\mathbf{b}\in[W^{1,\infty}(\Omega)]^2$. The functions $\mathbf{b}$, c, and f are assumed to be smooth on Ω. For any $(x,y)\in\overline{\Omega}$, assume that

$$b_1(x,y)\geq\beta_1>0,\quad b_2(x,y)\geq\beta_2>0,\quad c(x,y)+\frac{1}{2}\nabla\cdot\mathbf{b}(x,y)\geq\gamma>0, \tag{1.2}$$

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where β_1, β_2, and γ are some positive constants. Assumption (1.2) ensures that problem (1.1a)-(1.1b) has a unique solution in $H^2(\Omega)\cup H_0^1(\Omega)$ for all $f\in L^2(\Omega)$; see details in [17].

Singularly perturbed problems are an important topic in scientific computation. It is well known that the solution of a boundary value problem usually has layers, which are thin regions where the solution or its derivatives change rapidly due to a very small diffusion coefficient. To resolve the difficulty, numerical stabilization techniques have been developed, which can be divided into fitted operator methods and fitted mesh methods. One effective method for solving singularly perturbed problems is to use layer-adapted meshes. Boundary layers can be resolved by designing layer-adapted meshes if we have prior knowledge of the layer structure. Commonly used layer-adapted meshes for solving singularly perturbed problems include Bakhvalov-type and Shishkin-type meshes. Bakhvalov mesh was proposed for the first time in [2]; its application requires a nonlinear equation that cannot be solved explicitly. To avoid this difficulty, meshes that arise from an approximation of Bakhvalov's mesh generating function are called Bakhvalov-type meshes. These are among the most popular layer-adapted meshes; see details in [12]. Another piecewise equidistant mesh was proposed by Shishkin in [19], but a logarithmic factor will be present in the error bounds when using Shishkin-type mesh. Therefore, Bakhvalov-type meshes have better numerical performance than Shishkin-type meshes in general. Even with the use of layer-adapted meshes, the numerical solution of convection-dominated problems can still exhibit some oscillation, as detailed in [4]. Additional stabilization is added to the numerical scheme to address these oscillatory behaviors. Examples for singularly perturbed onvection-diffusion problems include the streamline-diffusion finite element method [13, 14], the classical finite difference method of up-winding flavor [1, 6, 11, 20] and the discontinuous Galerkin methods [7, 8, 18, 30, 35, 37].

In this paper, we consider the WG method to solve the singularly perturbed convection-diffusion boundary value problem on a Bakhvalov-type mesh. The WG method has proven to be an effective numerical technique for the partial differential equations (PDEs). The main idea of this method is to replace the classical derivative with a weak derivative, allowing the use of discontinuous functions in numerical schemes with parameter-independent stabilizers. The initial proposal for its application in solving second-order elliptic problems was made by Junping Wang and Xiu Ye in [25]. The WG method has been applied to various problems, including Stokes equations [26, 27, 29], Maxwell's equations [16], Brinkman equations [15, 28, 31], fractional time convection-diffusion problems [21] and others. For singular perturbed value problems, the WG method has also yielded results, as shown in [3, 9, 22–24, 33, 34, 36]. The main purpose of this paper is to present optimal order uniform convergence in the energy norm on a Bakhvalov-type mesh for convection-dominated problems in 2D.

This paper is organized as follows. In Section 2, we describe the assumptions and introduce a Bakhvalov-type mesh. In Section 3, we introduce the definition of the weak operator, the WG scheme, and some properties of the projection operator involved. In Section 4, we provide the convergence analysis. In Section 5, numerical results verify the correctness of our theory.

2 Assumption and partition

In this section, we will introduce the construction of the Bakhvalov-type mesh and the decomposition of the solution u. As in [10] we shall introduce the following assumption, which describes the structure of u.

Assumption 2.1 ([10])**.** For analysis, the solution u of Eq. (1.1a) can be decomposed as follows

$$u = S + E_1 + E_2 + E_{12},$$

where S represents the smooth part, E_1 and E_2 correspond to the boundary layer components, and E_{12} is the corner layer part. Then, for $0 \le i+j \le k+1$, there exists a constant C such that

$$\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}\right| \le C, \qquad \left|\frac{\partial^{i+j} E_1}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-j} e^{-\frac{\beta_2 y}{\varepsilon}}, \tag{2.1a}$$

$$\left|\frac{\partial^{i+j} E_2}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-i} e^{-\frac{\beta_1 x}{\varepsilon}}, \qquad \left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}\right| \le C\varepsilon^{-(i+j)} e^{-\frac{\beta_1 x + \beta_2 y}{\varepsilon}}. \tag{2.1b}$$

For the convection-diffusion problem (1.1a)-(1.1b), we consider the following Bakhvalov-type mesh introduced in [12], which is defined by

$$x_i = \begin{cases} -\dfrac{\sigma\varepsilon}{\beta_1}\ln(1-2(1-\varepsilon)i/N) & \text{for } i=0,\cdots,N/2, \\ 1-(1-x_{\frac{N}{2}})2(N-i)/N & \text{for } i=\frac{N}{2}+1,\cdots,N, \end{cases} \tag{2.2a}$$

$$y_j = \begin{cases} -\dfrac{\sigma\varepsilon}{\beta_1}\ln(1-2(1-\varepsilon)j/N) & \text{for } j=0,\cdots,N/2, \\ 1-(1-y_{\frac{N}{2}})2(N-j)/N & \text{for } j=\frac{N}{2}+1,\cdots,N, \end{cases} \tag{2.2b}$$

where $N \ge 4$ is a positive integer and $\sigma \ge k+1$, see Fig. 1. Transition points are $x_{N/2}$ and $y_{N/2}$, indicating a shift in mesh from coarse to fine. Then, we obtain a rectangulation mesh denoted as $\mathcal{T}_N$. For brevity, set

$$\Omega_{12} =: [x_0, x_{N/2-1}] \times [y_0, y_{N/2-1}], \qquad \Omega_1 =: [x_{N/2-1}, x_N] \times [y_0, y_{N/2-1}],$$
$$\Omega_2 =: [x_0, x_{N/2-1}] \times [y_{N/2-1}, y_N], \qquad \Omega_0 =: [x_{N/2-1}, x_N] \times [y_{N/2-1}, y_N].$$

Let $h_{x,i} = x_i - x_{i-1}$ and $h_{y,j} = y_j - y_{j-1}$. We omit the subscript x or y if there is no confusion. According to [32], we have the following lemma, which introduces some important properties of Bakhvalov-type mesh.

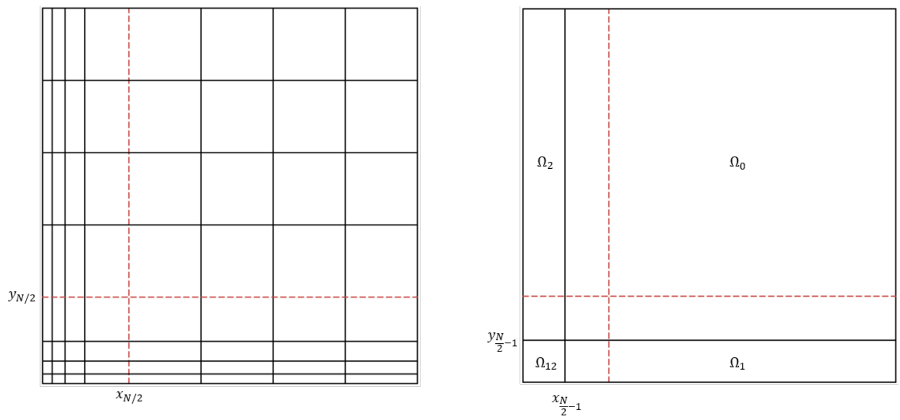

Figure 1: The Bakhvalov-type mesh with $N=8$ and dissection of Ω.

Lemma 2.1 ([32]). *In this paper, suppose that $\varepsilon \le N^{-1}$, then for the Bakhvalov-type mesh* (2.2), *one can obtain the properties*

$$C_1\varepsilon N^{-1} \le h_1 \le C_2\varepsilon N^{-1}, \tag{2.3a}$$
$$h_1 \le h_2 \le \cdots \le h_{N/2-1}, \tag{2.3b}$$
$$\frac{1}{4}\sigma\varepsilon \le h_{N/2-1} \le \sigma\varepsilon, \tag{2.3c}$$
$$\frac{1}{2}\sigma\varepsilon \le h_{N/2} \le 2\sigma N^{-1}, \tag{2.3d}$$
$$N^{-1} \le h_i \le 2N^{-1}, \qquad N/2+1 \le i \le N, \tag{2.3e}$$
$$C_3\sigma\varepsilon \ln N \le x_{N/2-1} \le C_4\sigma \ln N, \qquad x_{N/2} \ge C\sigma\varepsilon|\ln\varepsilon|, \tag{2.3f}$$
$$h_i^{\rho} e^{-\beta_1 x/\varepsilon} \le C\varepsilon^{\rho} N^{-\rho}, \qquad 1 \le i \le N/2-1, \quad 0 \le \rho \le \sigma. \tag{2.3g}$$

3 WG scheme

In this section, we introduce the notions of the WG method. Let $T \in \mathcal{T}_N$ be any element with boundary ∂T. We introduce a weak function $v=\{v_0,v_b\}$ on the element T, where $v_0 \in L^2(T)$ and $v_b \in L^2(\partial T)$. Note that v_b has a single value on each edge e. Additionally, component v_b may not necessarily be the same as the trace of v_0 on ∂T.

For any integer $k \ge 1$, we introduce the space of weak functions

$$V_N = \{v=\{v_0,v_b\} : v_0 \in \mathbb{Q}_k(T),\ v_b \in \mathbb{P}_k(e),\ e \in \partial T,\ \forall T \in \mathcal{T}_N\},$$

where $\mathbb{Q}_k$ is the space of polynomials of degree not exceeding k with respect to each of the variables x and y. Let V_N^0 represent the subspace of V_h defined by

$$V_N^0 = \{v \in V_N,\ v_b|_e = 0,\ e \subset \partial T \cap \partial\Omega\}.$$

Definition 3.1. *For any $v\in V_N$, a discrete weak gradient ∇_w is defined on T as a unique polynomial $\nabla_w v\in[\mathbb{Q}_k(T)]^2$ that satisfies*

$$(\nabla_w v,\mathbf{q})_T=-(v_0,\nabla\cdot\mathbf{q})_T+\langle v_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T},\quad \forall\mathbf{q}\in[\mathbb{Q}_k(T)]^2. \tag{3.1}$$

Definition 3.2. *For any $v\in V_N$, a discrete weak convection divergence $\nabla_w^b v\in\mathbb{Q}_k(T)$, related to $\mathbf{b}$, is defined on T as a unique polynomial satisfying*

$$\left(\nabla_{w,k}^b v,\varphi\right)_T=-(v_0,\nabla\cdot(\mathbf{b}\varphi))_T+\langle v_b,\mathbf{b}\cdot\mathbf{n}\varphi\rangle_{\partial T},\quad \forall\varphi\in\mathbb{Q}_k(T), \tag{3.2}$$

where $\mathbf{n}$ is the outward normal direction to ∂T.

The following notations are often used

$$(\varphi,\phi)_{\mathcal{T}_N}=\sum_{T\in\mathcal{T}_N}(\varphi,\phi)_T,\langle\varphi,\phi\rangle_{\partial\mathcal{T}_N}=\sum_{T\in\mathcal{T}_N}\langle\varphi,\phi\rangle_{\partial T},\|\varphi\|_{\mathcal{T}_N}^2=\sum_{T\in\mathcal{T}_N}\|\varphi\|_T^2.$$

We define ∂T_x and ∂T_y as the sets of element edges that are parallel to the x and y axes, respectively. Let

$$\partial_+T=\{(x,y)\in\partial T|\mathbf{b}(x,y)\cdot\mathbf{n}(x,y)\le 0\}.$$

The ϑ_T is the penalization parameter given by

$$\vartheta_T=\begin{cases}N, & \text{if } T\in\Omega_0,\\ \varepsilon h_y^{-1}, & \text{if } T\in\Omega\backslash\Omega_0 \quad \text{on } \partial T_x,\\ \varepsilon h_x^{-1}, & \text{if } T\in\Omega\backslash\Omega_0 \quad \text{on } \partial T_y.\end{cases}$$

Then, a WG algorithm is proposed below.

Algorithm 3.1. Find an approximate solution $u_N=\{u_0,u_b\}\in V_N^0$ that satisfies

$$A(u_N,v)=(f,v_0),\quad \forall v\in V_N^0, \tag{3.3}$$

where

$$\begin{aligned}
&A(u_N,v)=A_d(u_N,v)+A_c(u_N,v),\\
&A_d(u_N,v)=\varepsilon(\nabla_w u_N,\nabla_w v)_{\mathcal{T}_N}+s_d(u_N,v),\\
&A_c(u_N,v)=-(\nabla_w^b u_N,v_0)_{\mathcal{T}_N}+(cu_0,v_0)_{\mathcal{T}_N}+s_c(u_N,v),\\
&s_d(u_N,v)=\sum_{i=x,y}\sum_{T\in\mathcal{T}_N}\vartheta_T\langle u_0-u_b,v_0-v_b\rangle_{\partial T_i},\\
&s_c(u_N,v)=\sum_{\substack{T\in\mathcal{T}_N\\ e\in\partial_+T}}\langle-\mathbf{b}\cdot\mathbf{n}(u_0-u_b),v_0-v_b\rangle_e,
\end{aligned}$$

From the bilinear form in (3.3), we define an energy norm $|||\cdot|||$ in V_N^0.

Definition 3.3. *For any $v\in V_N^0$, we can derive an energy norm on V_N^0 defined by*

$$|||v|||^2=A_d(v,v)+\|v_0\|_{\mathcal{T}_N}+\||\mathbf{b}\cdot\mathbf{n}|^{\frac{1}{2}}(v_0-v_b)\|_{\partial\mathcal{T}_N}. \tag{3.4}$$

The following lemma deals with the coercivity of the bilinear form defined in (3.3).

Lemma 3.1. *There exists a positive constant α, independent of ε, such that for $v\in V_N^0$*

$$A(v,v)\geq\alpha|||v|||^2, \tag{3.5}$$

where $\alpha=\min\{\gamma,\frac{1}{2}\}$.

Proof. For any $v\in V_N^0$, it follows from the definition of the weak divergence (3.2) that

$$\begin{aligned}
(\nabla_w^b v,v_0)_{\mathcal{T}_N}&=-(v_0,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}+\langle v_b,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}\\
&=-\frac{1}{2}(v_0,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}-\frac{1}{2}(v_0,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}+\langle v_b,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}\\
&=-\frac{1}{2}(v_0,\nabla\cdot\mathbf{b}v_0)_{\mathcal{T}_N}-\frac{1}{2}\langle v_0-v_b,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}+\frac{1}{2}\langle v_b,\mathbf{b}\cdot\mathbf{n}(v_0-v_b)\rangle_{\partial\mathcal{T}_N}\\
&=-\frac{1}{2}(v_0,\nabla\cdot\mathbf{b}v_0)_{\mathcal{T}_N}-\frac{1}{2}\langle\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial\mathcal{T}_N}.
\end{aligned} \tag{3.6}$$

Then, together with (1.2) and (3.6) we have

$$\begin{aligned}
A_c(v,v)&=-(\nabla_w^b v,v_0)_{\mathcal{T}_N}+(cv_0,v_0)_{\mathcal{T}_N}+\langle-\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial_+\mathcal{T}_N}\\
&=\left(\left(c+\frac{1}{2}\nabla\cdot\mathbf{b}\right)v_0,v_0\right)_{\mathcal{T}_N}+\frac{1}{2}\langle\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial\mathcal{T}_N}\\
&\quad+\langle-\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial_+\mathcal{T}_N}\\
&\geq(\gamma v_0,v_0)_{\mathcal{T}_N}+\frac{1}{2}\langle-\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial_+\mathcal{T}_N}\\
&\quad+\frac{1}{2}\langle\mathbf{b}\cdot\mathbf{n}(v_0-v_b),v_0-v_b\rangle_{\partial\mathcal{T}_N\backslash\partial_+\mathcal{T}_N}\\
&\geq(\gamma v_0,v_0)_{\mathcal{T}_N}+\frac{1}{2}\langle|\mathbf{b}\cdot\mathbf{n}|(v_0-v_b),v_0-v_b\rangle_{\partial\mathcal{T}_N}.
\end{aligned}$$

The above inequality and $A_d(v,v)$ yield

$$A(v,v)\geq\alpha|||v|||^2,$$

where $\alpha=\min\{\gamma,\frac{1}{2}\}$. Thus, the lemma is proved. □

Now, we introduce some L^2 projection operators for each element $T\in\mathcal{T}_N$, detailed as follows. Let the operator $\mathcal{Q}_0$ project onto $\mathbb{Q}_k(T)$. For each edge $e\in\partial T$, consider the operator $\mathcal{Q}_b$ which projects onto $\mathbb{P}_k(e)$. Finally, we define a projection operator $\mathcal{Q}_N u=\{\mathcal{Q}_0 u,\mathcal{Q}_b u\}$ for the solution u onto the space V_N on each element T. Additionally, denote by $\mathbf{Q}_N$ the projection operator onto $[\mathbb{Q}_k(T)]^2$. The following lemma addresses the commutative property of the weak gradient.

Lemma 3.2. *On each element $T\in\mathcal{T}_N$, for any $\phi\in H^1(T)$,*

$$\nabla_w \mathcal{Q}_N\phi=\mathbf{Q}_N\nabla\phi. \tag{3.7}$$

Proof. For any $\mathbf{q}\in[\mathbb{Q}_k(T)]^2$, from definition of the weak gradient (3.1), we obtain

$$\begin{aligned}(\nabla_w \mathcal{Q}_N\phi,\mathbf{q})_T&=-(\mathcal{Q}_0\phi,\nabla\cdot\mathbf{q})_T+\langle \mathcal{Q}_b\phi,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\&=-(\phi,\nabla\cdot\mathbf{q})_T+\langle\phi,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}\\&=(\nabla\phi,\mathbf{q})_T=(\mathbf{Q}_N\nabla\phi,\mathbf{q})_T.\end{aligned}$$

This corresponds to the equality (3.7). □

4 Error estimate

In this section, we will derive an error estimate for the WG finite element approximation u_N. Next, we derive the following error equation that the error satisfies.

Lemma 4.1. *Suppose u is the solution of* (1.1a)-(1.1b) *and u_N is the solution of* (3.3). *Denote $e_N=\mathcal{Q}_Nu-u_N$. Then for any $v\in V_N^0$, we have*

$$A(e_N,v)=l_1(u,v)-l_2(u,v)-l_3(u,v)+s_d(\mathcal{Q}_Nu,v)+s_c(\mathcal{Q}_Nu,v), \tag{4.1}$$

where

$$\begin{aligned}&l_1(u,v)=(\mathcal{Q}_0u-u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N},\\&l_2(u,v)=\langle\mathcal{Q}_bu-u,\mathbf{b}\cdot\mathbf{n}(v_0-v_b)\rangle_{\partial\mathcal{T}_N},\\&l_3(u,v)=\varepsilon\langle(\mathbf{Q}_N\nabla u-\nabla u)\cdot\mathbf{n},v_0-v_b\rangle_{\partial\mathcal{T}_N}.\end{aligned}$$

Proof. Using (3.1), (3.7) and integration by parts, we obtain

$$\begin{aligned}&(\nabla_w\mathcal{Q}_Nu,\nabla_wv)_{\mathcal{T}_N}=(\mathbf{Q}_N\nabla u,\nabla_wv)_{\mathcal{T}_N}\\=&-(v_0,\nabla\cdot(\mathbf{Q}_N\nabla u))_{\mathcal{T}_N}+\langle v_b,(\mathbf{Q}_N\nabla u)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_N}\\=&(\nabla v_0,\mathbf{Q}_N\nabla u)_{\mathcal{T}_N}-\langle v_0-v_b,(\mathbf{Q}_N\nabla u)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_N}\\=&(\nabla u,\nabla v_0)_{\mathcal{T}_N}-\langle v_0-v_b,(\mathbf{Q}_N\nabla u)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_N}\\=&(-\Delta u,v_0)_{\mathcal{T}_N}+\langle\nabla u\cdot\mathbf{n},v_0\rangle_{\partial\mathcal{T}_N}-\langle v_0-v_b,(\mathbf{Q}_N\nabla u)\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_N}-\langle\nabla u\cdot\mathbf{n},v_b\rangle_{\partial\mathcal{T}_N}\\=&(-\Delta u,v_0)_{\mathcal{T}_N}-\langle(\mathbf{Q}_N\nabla u-\nabla u)\cdot\mathbf{n},v_0-v_b\rangle_{\partial\mathcal{T}_N},\end{aligned}$$

where we utilized the fact that $\langle\nabla u\cdot\mathbf{n},v_b\rangle_{\partial\mathcal{T}_N}=0$. Similarly, from (3.2) and integration by parts, one obtains

$$\begin{aligned}&-(\nabla_w^b\mathcal{Q}_Nu,v_0)_{\mathcal{T}_N}\\=&(\mathcal{Q}_0u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}-\langle\mathcal{Q}_bu,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}\\=&(\mathcal{Q}_0u-u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}+(u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}-\langle\mathcal{Q}_bu,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}\\=&(\mathcal{Q}_0u-u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}-(\mathbf{b}\cdot\nabla u,v_0)_{\mathcal{T}_N}-\langle\mathcal{Q}_bu-u,\mathbf{b}\cdot\mathbf{n}v_0\rangle_{\partial\mathcal{T}_N}\\=&-(\mathbf{b}\cdot\nabla u,v_0)_{\mathcal{T}_N}+(\mathcal{Q}_0u-u,\nabla\cdot(\mathbf{b}v_0))_{\mathcal{T}_N}-\langle\mathcal{Q}_bu-u,\mathbf{b}\cdot\mathbf{n}(v_0-v_b)\rangle_{\partial\mathcal{T}_N},\end{aligned}$$

where we have used $\langle \mathcal{Q}_b u - u, \mathbf{b}\cdot\mathbf{n} v_b\rangle_{\partial\mathcal{T}_N} = 0$. Additionally, we have

$$(c\mathcal{Q}_N u, v_0)_{\mathcal{T}_N} = (c\mathcal{Q}_0 u, v_0)_{\mathcal{T}_N} = (cu, v_0)_{\mathcal{T}_N}.$$

Noting that

$$\varepsilon(\nabla_w u_N, \nabla_w v) - (\nabla_w^b u_N, v_0) + (cu_N, v_0) + s_d(u_N, v) + s_c(u_N, v) = (f, v_0),$$

and

$$-\varepsilon(\Delta u, v_0) - (\mathbf{b}\cdot\nabla u, v_0) + (cu, v_0) = (f, v_0).$$

Combined with the aforementioned equality, we deduce

$$\varepsilon(\nabla_w \mathcal{Q}_N u, \nabla_w v) - (\nabla_w^b \mathcal{Q}_N u, v_0) + (c\mathcal{Q}_N u, v_0) = (f, v_0) + l_1(u,v) - l_2(u,v) - l_3(u,v).$$

By adding $s_d(\mathcal{Q}_N u, v)$ and $s_c(\mathcal{Q}_N u, v)$ to both sides of the above equation, we obtain

$$A(\mathcal{Q}_N u - u_N, v) = l_1(u,v) - l_2(u,v) - l_3(u,v) + s_d(\mathcal{Q}_N u, v) + s_c(\mathcal{Q}_N u, v).$$

Thus, the lemma is proved. $\square$

The error equation (4.1) can be used to derive the following error estimate for the WG finite element solution.

Theorem 4.1. *Suppose that $u \in H^{k+1}(\Omega)$ and u_N be the solution of* (1.1a)-(1.1b) *and* (3.3), *respectively. Then we have*

$$|||\mathcal{Q}_N u - u_N||| \le CN^{-k}. \tag{4.2}$$

Proof. Let $v = e_N$ in the error equation (4.1), we derive the following equation

$$A(e_N, e_N) = l_1(u, e_N) - l_2(u, e_N) - l_3(u, e_N) + s_d(\mathcal{Q}_N u, e_N) + s_c(\mathcal{Q}_N u, e_N).$$

Using the Cauchy-Schwarz inequality and (A.3a), we obtain

$$\begin{aligned}
l_1(u, e_N) &= (\mathcal{Q}_0 u - u, \nabla\cdot(\mathbf{b} e_0))_{\mathcal{T}_N} \\
&= (\mathcal{Q}_0 u - u, \nabla\cdot\mathbf{b} e_0)_{\mathcal{T}_N} + (\mathcal{Q}_0 u - u, \mathbf{b}\cdot\nabla e_0)_{\mathcal{T}_N} \\
&\le C\|\mathcal{Q}_0 u - u\|_{\mathcal{T}_N}\|e_0\|_{\mathcal{T}_N} + \Big(\mathcal{Q}_0 u - u, \Big(b_1 - \overline{b_1}\Big)\partial_x e_0 + \Big(b_2 - \overline{b_2}\Big)\partial_y e_0\Big)_{\mathcal{T}_N} \\
&\le C\|\mathcal{Q}_0 u - u\|_{\mathcal{T}_N}\|e_0\|_{\mathcal{T}_N} \\
&\le CN^{-(k+1)}|||e_N|||,
\end{aligned}$$

where $\overline{b_1}$ and $\overline{b_2}$ are piecewise constant functions with values $h_x^{-1}\int_e b_1 dx$ and $h_y^{-1}\int_e b_2 dy$, respectively. Similarly, from the Cauchy-Schwarz inequality and (A.3b), it follows that

$$
\begin{aligned}
l_2(u,e_N) =& \langle \mathcal{Q}_b u - u, \mathbf{b}\cdot\mathbf{n}(e_0-e_b)\rangle_{\partial\mathcal{T}_N}\\
\leq& C\sum_{i=x,y}\sum_{T\in\mathcal{T}_N}\|\mathcal{Q}_0 u-u\|_{\partial T_i}\left\||\mathbf{b}\cdot\mathbf{n}|^{\frac{1}{2}}(e_0-e_b)\right\|_{\partial T_i}\\
\leq& C\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\|\mathcal{Q}_0 u-u\|^2_{\partial T_i}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_N}\left\||\mathbf{b}\cdot\mathbf{n}|^{\frac{1}{2}}(e_0-e_b)\right\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\
\leq& CN^{-(k+\frac{1}{2})}|||e_N|||.
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality and (A.3c), we obtain

$$
\begin{aligned}
l_3(u,e_N) =& \varepsilon\langle(\mathbf{Q}_N\nabla u-\nabla u)\cdot\mathbf{n}, e_0-e_b\rangle_{\partial\mathcal{T}_N}\\
\leq& C\varepsilon\sum_{i=x,y}\sum_{T\in\mathcal{T}_N}\|\mathbf{Q}_N\nabla u-\nabla u\|_{\partial T_i}\|e_0-e_b\|_{\partial T_i}\\
\leq& C\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\theta_T\|\mathbf{Q}_N\nabla u-\nabla u\|^2_{\partial T_i}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_N}\vartheta_T\|e_0-e_b\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\
\leq& CN^{-k}|||e_N|||.
\end{aligned}
$$

Furthermore, using the definition of stabilization, the Cauchy-Schwarz inequality (A.3d), and (A.3b), we derive

$$
\begin{aligned}
s_d(\mathcal{Q}_N u,e_N) =& \sum_{i=x,y}\sum_{T\in\mathcal{T}_N}\vartheta_T\langle\mathcal{Q}_0 u-\mathcal{Q}_b u, e_0-e_b\rangle_{\partial T_i}\\
\leq& C\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\vartheta_T\|\mathcal{Q}_0 u-u\|^2_{\partial T_i}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_N}\vartheta_T\|e_0-e_b\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\
\leq& CN^{-k}|||e_N|||,
\end{aligned}
$$

and

$$
\begin{aligned}
s_c(\mathcal{Q}_N u,e_N) =& \sum_{i=x,y}\sum_{T\in\mathcal{T}_N}\langle-\mathbf{b}\cdot\mathbf{n}(\mathcal{Q}_0 u-\mathcal{Q}_b u), v_0-v_b\rangle_{\partial_+ T_i}\\
\leq& C\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\|\mathcal{Q}_0 u-u\|^2_{\partial T_i}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}_N}|\mathbf{b}\cdot\mathbf{n}|^{\frac{1}{2}}\|e_0-e_b\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\
\leq& CN^{-(k+\frac{1}{2})}|||e_N|||.
\end{aligned}
$$

Combining all these estimates with (3.5), we deduce

$$\begin{aligned}\alpha|||e_N|||^2 \le & A(e_N,e_N)\\ \le & |l_1(u,e_N)|+|l_2(u,e_N)|+|l_3(u,e_N)|+|s_d(\mathcal{Q}_N u,e_N)|+|s_c(\mathcal{Q}_N u,e_N)|\\ \le & CN^{-k}|||e_N|||,\end{aligned}$$

which implies (4.2). This completes the proof of the theorem. □

5 Numerical experiments

In this section we present numerical experiments that support our theoretical results. The solution exhibits typical exponential layer behavior. And we select $\sigma=2k$ for creating the Bakhvalov-type mesh.

Example 5.1. For

$$\begin{aligned}&-\varepsilon\Delta u-(2+2x-y)u_x-(3-x+2y)u_y+u=f && \text{in } \Omega=(0,1)^2,\\ &u=0 && \text{on } \partial\Omega,\end{aligned}$$

where $f(x,y)$ is chosen such that the exact solution

$$u(x,y)=2\sin(\pi x)\left(1-e^{-\frac{2x}{\varepsilon}}\right)(1-y)^2\left(1-e^{-\frac{y}{\varepsilon}}\right).$$

The error $|||e_N|||$ and convergence rates for several different ε are shown in Tables 1-2.

Example 5.2. For

$$\begin{aligned}&-\varepsilon\Delta u-2u_x-3u_y+u=f && \text{in } \Omega=(0,1)^2,\\ &u=0 && \text{on } \partial\Omega,\end{aligned}$$

where $f(x,y)$ is chosen such that the exact solution

$$u(x,y)=2\sin(1-x)\left(1-e^{-\frac{2x}{\varepsilon}}\right)(1-y)^2\left(1-e^{-\frac{3y}{\varepsilon}}\right).$$

The error $|||e_N|||$ and convergence rates for several different ε are shown in Tables 3-4.

Table 1: Example 5.1, $k=1$.

N	$\varepsilon=$ 1e-6		$\varepsilon=$ 1e-7		$\varepsilon=$ 1e-8		$\varepsilon=$ 1e-9		$\varepsilon=$ 1e-10	
8	3.66E-01	0.00	3.84E-01	0.00	3.98E-01	0.00	4.07E-01	0.00	4.14E-01	0.00
16	1.73E-01	1.08	1.83E-01	1.07	1.91E-01	1.06	1.96E-01	1.05	2.00E-01	1.05
32	8.24E-02	1.07	8.81E-02	1.05	9.23E-02	1.05	9.54E-02	1.04	9.76E-02	1.03
64	3.95E-02	1.06	4.27E-02	1.04	4.51E-02	1.03	4.68E-02	1.03	4.80E-02	1.02
128	1.90E-02	1.06	2.08E-02	1.04	2.21E-02	1.03	2.30E-02	1.02	2.37E-02	1.02
256	9.14E-03	1.06	1.01E-02	1.04	1.08E-02	1.03	1.14E-02	1.02	1.17E-02	1.01

Table 2: Example 5.1, $k=2$.

N	$\varepsilon=$ 1e-6		$\varepsilon=$ 1e-7		$\varepsilon=$ 1e-8		$\varepsilon=$ 1e-9		$\varepsilon=$ 1e-10	
8	7.89E-02	0.00	8.27E-02	0.00	8.52E-02	0.00	8.70E-02	0.00	8.85E-02	0.00
16	1.93E-02	2.03	2.04E-02	2.02	2.11E-02	2.02	2.16E-02	2.01	2.20E-02	2.01
32	4.70E-03	2.04	5.01E-03	2.02	5.21E-03	2.02	5.35E-03	2.01	5.45E-03	2.01
64	1.14E-03	2.04	1.23E-03	2.02	1.29E-03	2.02	1.33E-03	2.01	1.35E-03	2.01
128	2.77E-04	2.05	3.02E-04	2.03	3.18E-04	2.02	3.29E-04	2.01	3.36E-04	2.01

Table 3: Example 5.2, $k=1$.

N	$\varepsilon=$ 1e-6		$\varepsilon=$ 1e-7		$\varepsilon=$ 1e-8		$\varepsilon=$ 1e-9		$\varepsilon=$ 1e-10	
8	8.67E-02	0.00	8.68E-02	0.00	8.69E-02	0.00	8.70E-02	0.00	8.71E-02	0.00
16	3.53E-02	1.30	3.53E-02	1.30	3.54E-02	1.30	3.54E-02	1.30	3.54E-02	1.30
32	1.50E-02	1.24	1.50E-02	1.24	1.50E-02	1.24	1.50E-02	1.24	1.50E-02	1.24
64	6.69E-03	1.16	6.69E-03	1.16	6.69E-03	1.16	6.69E-03	1.16	6.69E-03	1.16
128	3.14E-03	1.09	3.14E-03	1.09	3.14E-03	1.09	3.14E-03	1.09	3.14E-03	1.09
256	1.51E-03	1.05	1.51E-03	1.05	1.51E-03	1.05	1.51E-03	1.05	1.51E-03	1.05

Table 4: Example 5.2, $k=2$.

N	$\varepsilon=$ 1e-6		$\varepsilon=$ 1e-7		$\varepsilon=$ 1e-8		$\varepsilon=$ 1e-9		$\varepsilon=$ 1e-10	
8	1.58E-02	0.00	1.58E-02	0.00	1.58E-02	0.00	1.58E-02	0.00	1.58E-02	0.00
16	2.91E-03	2.44	2.91E-03	2.44	2.91E-03	2.44	2.91E-03	2.44	2.91E-03	2.44
32	5.83E-04	2.32	5.83E-04	2.32	5.83E-04	2.32	5.83E-04	2.32	5.83E-04	2.32
64	1.29E-04	2.17	1.29E-04	2.17	1.29E-04	2.17	1.29E-04	2.17	1.29E-04	2.17
128	3.05E-05	2.08	3.05E-05	2.08	3.05E-05	2.08	3.05E-05	2.08	3.05E-05	2.08

Tables 1–4 present our numerical results, including the errors in the energy norm and convergence rates for Examples 5.1 and 5.2. It is evident that when the polynomial degree $k=1$ or 2, the convergence rate of the two numerical examples also reaches 1 or 2, respectively. These data show the uniform convergence with respect to the singular perturbation parameter ε.

6 Conclusions

In this paper, we introduce the weak Galerkin method for solving the singularly perturbed convection-diffusion problems on the Bakhvalov-type mesh. We construct the corresponding numerical scheme and develop the convergence analysis in the energy norm, obtaining the optimal convergence order. We present two numerical examples to validate the correctness of the convergence theory.

Appendix: some technique tools

The goal of this appendix is to establish fundamental estimates that are useful in the error estimate. The following lemmas are employed in the convergence analysis, and readers are referred to [5] for a detailed proof.

Lemma A.1 ([5])**.** *Consider $\phi \in H^{k+1}(T)$ with $k \geq 1$. Let $\mathcal{Q}_0\phi$ denote the L^2 projection of ϕ onto $\mathbb{Q}_k(T)$. Then the following inequalities estimate hold,*

$$\|\phi-\mathcal{Q}_0\phi\|_T \leq C\left(h_x^{k+1}\left\|\partial_x^{k+1}\phi\right\|_T + h_y^{k+1}\left\|\partial_y^{k+1}\phi\right\|_T\right), \tag{A.1a}$$

$$\|\phi-\mathcal{Q}_0\phi\|_{\partial T_i} \leq C\left(h_j^{k+\frac{1}{2}}\|\partial_j^{k+1}\phi\|_T + h_i^{k+1}h_j^{-\frac{1}{2}}\|\partial_i^{k+1}\phi\|_T + h_j^{\frac{1}{2}}h_i^k\|\partial_i^k\partial_j\phi\|_T\right), \tag{A.1b}$$

where $i,j \in \{x,y\}$ and $i \neq j$.

Lemma A.2 ([5])**.** *Let $v \in \mathbb{Q}_k(T)$ with $k \geq 1$ such that the following inequalities hold,*

$$\|v\|_{\partial T_i} \leq C h_j^{-\frac{1}{2}}\|v\|_T, \tag{A.2a}$$

$$\|\partial_i v\|_T \leq C h_i^{-1}\|v\|_T, \tag{A.2b}$$

where C is a constant that depends on k, and $i,j \in \{x,y\}$, $i \neq j$.

We aim to establish the following estimates, which are useful in the convergence analysis for the WG scheme (3.3).

Lemma A.3. *Let $k \geq 1$, $\sigma \geq k+1$, and $u \in H^{k+1}(\Omega)$. There exists a constant C such that the following estimates hold*

$$\left(\sum_{T\in\mathcal{T}_N}\|\mathcal{Q}_0u-u\|_T^2\right)^{\frac{1}{2}} \leq CN^{-(k+1)}, \tag{A.3a}$$

$$\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\|\mathcal{Q}_0u-u\|_{\partial T_i}^2\right)^{\frac{1}{2}} \leq CN^{-(k+\frac{1}{2})}, \tag{A.3b}$$

$$\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\theta_T\|\mathbf{Q}_N\nabla u-\nabla u\|_{\partial T_i}^2\right)^{\frac{1}{2}} \leq CN^{-k}, \tag{A.3c}$$

$$\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\vartheta_T\|\mathcal{Q}_0u-u\|_{\partial T_i}^2\right)^{\frac{1}{2}} \leq CN^{-k}, \tag{A.3d}$$

where

$$\theta_T=\begin{cases}\varepsilon^2N^{-1}, & \text{if } T\in\Omega_0,\\ \varepsilon h_y, & \text{if } T\in\Omega\backslash\Omega_0 \quad \text{on } \partial T_x,\\ \varepsilon h_x, & \text{if } T\in\Omega\backslash\Omega_0 \quad \text{on } \partial T_y.\end{cases}$$

Proof. Each term in the Assumption 2.1 will be considered individually. Let Ω_ℓ represent any region in Ω_0, Ω_1, Ω_2 and Ω_{12}. To derive inequality (A.3a), we use (A.1a) and Lemma 2.1 to obtain

$$
\begin{aligned}
\left(\sum_{T\in\Omega_\ell}\|S-\mathcal{Q}_0S\|_T^2\right)^{\frac{1}{2}} \le & C\left(\sum_{T\in\Omega_\ell}\left(h_x^{2(k+1)}+h_y^{2(k+1)}\right)\cdot h_xh_y\right)^{\frac{1}{2}} \\
\le & CN\left(h_N^{2(k+2)}\right)^{\frac{1}{2}}\le CN^{-(k+1)}.
\end{aligned}
$$

Next, considering the boundary layer E_1 in region $\Omega_{12}\cup\Omega_1$, using (A.1a) and Lemma 2.1 we obtain

$$
\begin{aligned}
\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\|\mathcal{Q}_0E_1-E_1\|_T^2\right)^{\frac{1}{2}} \le & C\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(h_x^{2(k+1)}\left\|\partial_x^{k+1}E_1\right\|_T^2+h_y^{2(k+1)}\left\|\partial_y^{k+1}E_1\right\|_T^2\right)\right)^{\frac{1}{2}} \\
\le & C\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(h_x^{2k+3}h_y+\varepsilon^{-2(k+1)}h_xh_y^{2k+3}\right)\left\|e^{-\frac{\beta_2y}{\varepsilon}}\right\|_{L^\infty(T)}^2\right)^{\frac{1}{2}} \\
\le & CN\left(h_{x,N}^{2k+3}\varepsilon N^{-1}+h_{x,N}h_{y,\frac{N}{2}-1}N^{-2(k+1)}\right)^{\frac{1}{2}} \\
\le & CN\left(\varepsilon N^{-2(k+2)}+\varepsilon N^{-(2k+3)}\right)^{\frac{1}{2}}\le CN^{-(k+1)}.
\end{aligned}
$$

For the region $\Omega_2\cup\Omega_0$, we have the following estimate

$$
\begin{aligned}
\left(\sum_{T\in\Omega_2\cup\Omega_0}\|\mathcal{Q}_0E_1-E_1\|_T^2\right)^{\frac{1}{2}} \le & C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_1\|_T^2+\|\mathcal{Q}_0E_1\|_T^2\right)\right)^{\frac{1}{2}}\le C\|E_1\|_{\Omega_2\cup\Omega_0} \\
\le & CN\left(h_{x,N}h_{y,N}\|E_1\|_{L^\infty(\Omega_2\cup\Omega_0)}^2\right)^{\frac{1}{2}}\le CN^{-\sigma}.
\end{aligned}
$$

A similar bound can be readily obtained for E_2. For the concer layer E_{12}, by applying inequality (A.1a) and Lemma 2.1, we obtain

$$
\begin{aligned}
\left(\sum_{T\in\Omega_{12}}\|\mathcal{Q}_0E_{12}-E_{12}\|_T^2\right)^{\frac{1}{2}} \le & C\left(\sum_{T\in\Omega_{12}}\left(h_x^{2k+3}h_y\left\|\partial_x^{k+1}E_{12}\right\|_{L^\infty(T)}^2+h_xh_y^{2k+3}\left\|\partial_y^{k+1}E_{12}\right\|_{L^\infty(T)}^2\right)\right)^{\frac{1}{2}} \\
\le & C\left(\sum_{T\in\Omega_{12}}\varepsilon^{-2(k+1)}\left(h_x^{2k+3}h_y+h_xh_y^{2k+3}\right)\left\|e^{-\frac{\beta_1x+\beta_2y}{\varepsilon}}\right\|_{L^\infty(T)}^2\right)^{\frac{1}{2}} \\
\le & CN\left(\varepsilon N^{-1}N^{-2(k+1)}\left(h_{x,\frac{N}{2}-1}+h_{y,\frac{N}{2}-1}\right)\right)^{\frac{1}{2}} \\
\le & CN\left(\varepsilon^2N^{-(2k+3)}\right)^{\frac{1}{2}}\le CN^{-(k+\frac{3}{2})}.
\end{aligned}
$$

Additionally, we have the estimate in regions $\Omega_1\cup\Omega_0$ and $\Omega_2\cup\Omega_0$ as follows

$$\begin{aligned}\left(\sum_{T\in\Omega_1\cup\Omega_0}\|\mathcal{Q}_0E_{12}-E_{12}\|_T^2\right)^{\frac{1}{2}}\le&C\left(\sum_{T\in\Omega_1\cup\Omega_0}\left(\|E_{12}\|_T^2+\|\mathcal{Q}_0E_{12}\|_T^2\right)\right)^{\frac{1}{2}}\le C\|E_{12}\|_{\Omega_1\cup\Omega_0}\\ \le&CN\left(h_{x,N}h_{y,N}\|E_{12}\|^2_{L^\infty(\Omega_1\cup\Omega_0)}\right)^{\frac{1}{2}}\le CN^{-\sigma},\end{aligned}$$

and

$$\begin{aligned}\left(\sum_{T\in\Omega_2\cup\Omega_0}\|\mathcal{Q}_0E_{12}-E_{12}\|_T^2\right)^{\frac{1}{2}}\le&C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_{12}\|_T^2+\|\mathcal{Q}_0E_{12}\|_T^2\right)\right)^{\frac{1}{2}}\le C\|E_{12}\|_{\Omega_2\cup\Omega_0}\\ \le&CN\left(h_{x,N}h_{y,N}\|E_{12}\|^2_{L^\infty(\Omega_2\cup\Omega_0)}\right)^{\frac{1}{2}}\le CN^{-\sigma}.\end{aligned}$$

Combining the above inequalities completes the proof of (A.3a).

Next, we provide the proof of (A.3b). For any region Ω_ℓ, by using (A.1b) and Lemma 2.1, we obtain

$$\begin{aligned}\left(\sum_{T\in\Omega_\ell}\|\mathcal{Q}_0S-S\|_{\partial T_i}^2\right)^{\frac{1}{2}}\le&C\left(\sum_{T\in\Omega_\ell}\left(h_j^{2k+1}+h_i^{2(k+1)}h_j^{-1}+h_i^{2k}h_j\right)\cdot h_ih_j\right)^{\frac{1}{2}}\\ \le&CN\left(h_N^{2k+1}\right)^{\frac{1}{2}}\le CN^{-(k+\frac{1}{2})}.\end{aligned}$$

Concerning the boundary layer E_1 in region $\Omega_{12}\cup\Omega_1$, using (A.1b) and Lemma 2.1, we have

$$\begin{aligned}&\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\|\mathcal{Q}_0E_1-E_1\|_{\partial T_x}^2\right)^{\frac{1}{2}}\\ \le&C\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(h_y^{2k+1}\left\|\partial_y^{k+1}E_1\right\|_T^2+h_x^{2(k+1)}h_y^{-1}\left\|\partial_x^{k+1}E_1\right\|_T^2+h_x^{2k}h_y\left\|\partial_x^k\partial_yE_1\right\|_T^2\right)\right)^{\frac{1}{2}}\\ \le&C\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(\varepsilon^{-2(k+1)}\cdot h_xh_y^{2(k+1)}+h_x^{2k+3}+\varepsilon^{-2}\cdot h_x^{2k+1}h_y^2\right)\left\|e^{-\frac{\beta_2y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\ \le&CN\left(h_{x,N}N^{-2(k+1)}+h_{x,N}^{2k+3}+h_{x,N}^{2k+1}N^{-2}\right)^{\frac{1}{2}}\le CN^{-(k+\frac{1}{2})}.\end{aligned}$$

Applying (A.2a), (A.2b), and Lemma 2.1 in the remaining regions yields

$$\begin{aligned}\left(\sum_{T\in\Omega_2\cup\Omega_0}\|\mathcal{Q}_0E_1-E_1\|_{\partial T_x}^2\right)^{\frac{1}{2}}\le&C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_1\|_{\partial T_x}^2+\|\mathcal{Q}_0E_1\|_{\partial T_x}^2\right)\right)^{\frac{1}{2}}\\ \le&C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_1\|_{\partial T_x}^2+h_y^{-1}\|E_1\|_T^2\right)\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\le C\left(N\int_0^1 e^{-\frac{2\beta_1 y}{\varepsilon}}dx+\sum_{T\in\Omega_2\cup\Omega_0}h_x\left\|e^{-\frac{\beta_2 y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
&\le C\left(N\cdot N^{-2\sigma}+h_{x,N}\cdot N^2\cdot N^{-2\sigma}\right)^{\frac{1}{2}}\le CN^{\frac{1}{2}-\sigma}.
\end{aligned}
$$

Similar results are easily obtained for ∂T_y and for the boundary layer E_2. Now, considering the concer layer E_{12}, we obtain

$$
\begin{aligned}
&\left(\sum_{T\in\Omega_{12}}\|\mathcal{Q}_0E_{12}-E_{12}\|^2_{\partial T_x}\right)^{\frac{1}{2}}\\
\le& C\left(\sum_{T\in\Omega_{12}}\left(h_y^{2k+1}\left\|\partial_y^{k+1}E_{12}\right\|^2_T+h_x^{2(k+1)}h_y^{-1}\left\|\partial_x^{k+1}E_{12}\right\|^2_T+h_x^{2k}h_y\left\|\partial_x^k\partial_yE_{12}\right\|^2_T\right)\right)^{\frac{1}{2}}\\
\le& C\left(\sum_{T\in\Omega_{12}}\varepsilon^{-2(k+1)}\left(h_xh_y^{2(k+1)}+h_x^{2k+3}+h_x^{2k+1}h_y^2\right)\left\|e^{-\frac{\beta_1x+\beta_2y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
\le& CN\left(\varepsilon N^{-1}N^{-2(k+1)}+h_{x,\frac{N}{2}-1}N^{-2(k+1)}+\varepsilon N^{-(2k+1)}N^{-2}\right)^{\frac{1}{2}}\le CN^{-(k+\frac{1}{2})},
\end{aligned}
$$

where (A.1b) and Lemma 2.1 are utilized. In other regions, using (A.2a), (A.2b), and Lemma 2.1, we derive

$$
\begin{aligned}
\left(\sum_{T\in\Omega_1\cup\Omega_0}\|\mathcal{Q}_0E_{12}-E_{12}\|^2_{\partial T_x}\right)^{\frac{1}{2}}\le& C\left(\sum_{T\in\Omega_1\cup\Omega_0}\left(\|E_{12}\|^2_{\partial T_x}+\|\mathcal{Q}_0E_{12}\|^2_{\partial T_x}\right)\right)^{\frac{1}{2}}\\
\le& C\left(\sum_{T\in\Omega_1\cup\Omega_0}\left(\|E_{12}\|^2_{\partial T_x}+h_y^{-1}\|E_{12}\|^2_T\right)\right)^{\frac{1}{2}}\\
\le& C\left(N\int_{x_{N/2-1}}^1 e^{-\frac{2(\beta_1x+\beta_2y)}{\varepsilon}}dx+\sum_{T\in\Omega_1\cup\Omega_0}h_x\left\|e^{-\frac{\beta_1x+\beta_2y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
\le& C\left(N\cdot\varepsilon N^{-2\sigma}+h_{x,N}\cdot N^2\cdot N^{-2\sigma}\right)^{\frac{1}{2}}\le CN^{\frac{1}{2}-\sigma},
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\sum_{T\in\Omega_2\cup\Omega_0}\|\mathcal{Q}_0E_{12}-E_{12}\|^2_{\partial T_x}\right)^{\frac{1}{2}}\le& C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_{12}\|^2_{\partial T_x}+\|\mathcal{Q}_0E_{12}\|^2_{\partial T_x}\right)\right)^{\frac{1}{2}}\\
\le& C\left(\sum_{T\in\Omega_2\cup\Omega_0}\left(\|E_{12}\|^2_{\partial T_x}+h_y^{-1}\|E_{12}\|^2_T\right)\right)^{\frac{1}{2}}\\
\le& C\left(N\int_0^1 e^{-\frac{2(\beta_1x+\beta_2y)}{\varepsilon}}dx+\sum_{T\in\Omega_2\cup\Omega_0}h_x\left\|e^{-\frac{\beta_1x+\beta_2y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C\left(N\cdot\varepsilon N^{-2\sigma}+h_{x,N}\cdot N^2\cdot N^{-2\sigma}\right)^{\frac{1}{2}}\leq CN^{\frac{1}{2}-\sigma}.$$

Similar results can be obtained for ∂T_y. Combining the above estimates, we establish inequality (A.3b).

Let us now prove (A.3c). The notation Ω_τ represents any region in Ω_{12}, Ω_1, and Ω_2. By the definition of θ_T and (A.1b), we obtain

$$\begin{aligned}\left(\sum_{T\in\Omega_\tau}\theta_T\|\mathbf{Q}_N\nabla S-\nabla S\|^2_{\partial T_i}\right)^{\frac{1}{2}}&=\left(\sum_{T\in\Omega_\tau}\varepsilon h_j\|\mathbf{Q}_N\nabla S-\nabla S\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\&\leq C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_\tau}\left(h_j^{2k}+h_i^{2k}+h_i^{2(k-1)}h_j^2\right)\cdot h_ih_j\right)^{\frac{1}{2}}\\&\leq C\varepsilon^{\frac{1}{2}}N\left(h_N^{2(k+1)}\right)^{\frac{1}{2}}\leq CN^{-(k+\frac{1}{2})}.\end{aligned}$$

Additionally, for the region Ω_0, we have

$$\begin{aligned}\left(\sum_{T\in\Omega_0}\theta_T\|\mathbf{Q}_N\nabla S-\nabla S\|^2_{\partial T_i}\right)^{\frac{1}{2}}&=\left(\sum_{T\in\Omega_0}\varepsilon^2N^{-1}\|\mathbf{Q}_N\nabla S-\nabla S\|^2_{\partial T_i}\right)^{\frac{1}{2}}\\&\leq C\varepsilon\left(\sum_{T\in\Omega_0}N^{-1}\left(h_j^{2k-1}+h_i^{2k}h_j^{-1}+h_i^{2(k-1)}h_j\right)\cdot h_ih_j\right)^{\frac{1}{2}}\\&\leq C\varepsilon N\left(N^{-1}\cdot h_N^{2k+1}\right)^{\frac{1}{2}}\leq CN^{-(k+\frac{1}{2})},\end{aligned}$$

where $i,j\in\{x,y\}$ and $i\neq j$. Considering the boundary layer E_1 in region $\Omega_{12}\cup\Omega_1$, we have

$$\begin{aligned}&\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\theta_T\|\mathbf{Q}_N\nabla E_1-\nabla E_1\|^2_{\partial T_x}\right)^{\frac{1}{2}}\\\leq& C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(h_x^{2(k-1)}h_y^2\left(\left\|\partial_x^k\partial_yE_1\right\|_T^2+\left\|\partial_x^{k-1}\partial_y^2E_1\right\|_T^2\right)\right.\right.\\&\left.\left.+h_y^{2k}\left(\left\|\partial_y^k\partial_xE_1\right\|_T^2+\left\|\partial_y^{k+1}E_1\right\|_T^2\right)+h_x^{2k}\left(\left\|\partial_x^{k+1}E_1\right\|_T^2+\left\|\partial_x^k\partial_yE_1\right\|_T^2\right)\right)\right)^{\frac{1}{2}}\\\leq& C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_{12}\cup\Omega_1}\left(h_y^{2k+1}h_x\left(\varepsilon^{-2k}+\varepsilon^{-2(k+1)}\right)+h_x^{2k+1}h_y\left(1+\varepsilon^{-2}\right)\right.\right.\\&\left.\left.+h_x^{2k-1}h_y^3\left(\varepsilon^{-2}+\varepsilon^{-4}\right)\right)\left\|e^{-\frac{\beta_2y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq C\varepsilon^{\frac{1}{2}} N\left(\left(h_{x,N}N^{-(2k+1)}+h_{x,N}^{2k+1}N^{-1}+h_{x,N}^{2k-1}N^{-3}\right)\cdot\left(\varepsilon+\varepsilon^{-1}\right)\right)^{\frac{1}{2}}\leq CN^{-k}.$$

In region Ω_2, using (A.2a), (A.2b), and Lemma 2.1 we obtain

$$\begin{aligned}
&\left(\sum_{T\in\Omega_2}\theta_T\|\mathbf{Q}_N\nabla E_1-\nabla E_1\|^2_{\partial T_x}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_2}h_y\left(\|\partial_x E_1\|^2_{\partial T_x}+\left\|\partial_y E_1\right\|^2_{\partial T_x}\right)+\|\partial_x E_1\|^2_T+\left\|\partial_y E_1\right\|^2_T\right)^{\frac{1}{2}}\\
\leq& C\varepsilon^{\frac{1}{2}}\left(h_{y,N}\left(1+\varepsilon^{-2}\right)N\int_0^{x_{\frac{N}{2}-1}}e^{-\frac{2\beta_2 y}{\varepsilon}}dx+\sum_{T\in\Omega_2}\left(1+\varepsilon^{-2}\right)h_xh_y\left\|e^{-\frac{\beta_2 y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon^{\frac{1}{2}}\left(\left(1+\varepsilon^{-2}\right)\left(\varepsilon\cdot N^{1-2\sigma}+\varepsilon N^{-1}\cdot N^{2-2\sigma}\right)\right)^{\frac{1}{2}}\leq CN^{\frac{1}{2}-\sigma}.
\end{aligned}$$

Similarly, for the region Ω_0, we have

$$\begin{aligned}
&\left(\sum_{T\in\Omega_0}\theta_T\|\mathbf{Q}_N\nabla E_1-\nabla E_1\|^2_{\partial T_x}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon\left(\sum_{T\in\Omega_0}N^{-1}\left(\|\partial_x E_1\|^2_{\partial T_x}+\left\|\partial_y E_1\right\|^2_{\partial T_x}\right)+N^{-1}h_y^{-1}\left(\|\partial_x E_1\|^2_T+\left\|\partial_y E_1\right\|^2_T\right)\right)^{\frac{1}{2}}\\
\leq& C\varepsilon\left(N^{-1}\left(1+\varepsilon^{-2}\right)N\int_{x_{N/2-1}}^1 e^{-\frac{2\beta_2 y}{\varepsilon}}dx+\sum_{T\in\Omega_0}\left(1+\varepsilon^{-2}\right)N^{-1}h_x\left\|e^{-\frac{\beta_2 y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon\left(\left(1+\varepsilon^{-2}\right)N^{-2\sigma}\right)^{\frac{1}{2}}\leq CN^{-\sigma}.
\end{aligned}$$

Similar results can be obtained for ∂T_y and the boundary layer E_2 in a similar manner. Applying arguments analogous to those used above, for the concer layer E_{12}, one has

$$\begin{aligned}
&\left(\sum_{T\in\Omega_{12}}\theta_T\|\mathbf{Q}_N\nabla E_{12}-\nabla E_{12}\|^2_{\partial T_x}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_{12}}\varepsilon^{-2(k+1)}\left(h_y^{2k+1}h_x+h_x^{2k+1}h_y+h_x^{2k-1}h_y^3\right)\left\|e^{-\frac{\beta_1 x+\beta_2 y}{\varepsilon}}\right\|^2_{L^\infty(T)}\right)^{\frac{1}{2}}\\
\leq& C\varepsilon^{\frac{1}{2}}N\left(N^{-(2k+1)}\cdot N^{-1}+N^{-(2k+1)}N^{-1}+N^{-(2k-1)}N^{-3}\right)^{\frac{1}{2}}\leq C\varepsilon^{\frac{1}{2}}N^{-k}.
\end{aligned}$$

Using (A.2a), (A.2b), and Lemma 2.1, we obtain

$$\begin{aligned}
&\left(\sum_{T\in\Omega_1}\theta_T\|\mathbf{Q}_N\nabla E_{12}-\nabla E_{12}\|_{\partial T_x}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_1}h_y\left(\|\partial_x E_{12}\|_{\partial T_x}^2+\left\|\partial_y E_{12}\right\|_{\partial T_x}^2\right)+\|\partial_x E_{12}\|_T^2+\left\|\partial_y E_{12}\right\|_T^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(h_{y,\frac{N}{2}-1}N\varepsilon^{-2}\int_{x_{\frac{N}{2}-1}}^1 e^{-\frac{2(\beta_1 x+\beta_2 y)}{\varepsilon}}dx+\sum_{T\in\Omega_1}\varepsilon^{-2}h_x h_y\left\|e^{-\frac{\beta_1 x+\beta_2 y}{\varepsilon}}\right\|_{L^\infty(T)}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(\varepsilon^{-2}\left(\varepsilon^2\cdot N^{1-2\sigma}+\varepsilon N^{-1}\cdot N^{2-2\sigma}\right)\right)^{\frac{1}{2}}\leq CN^{\frac{1}{2}-\sigma}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\left(\sum_{T\in\Omega_2}\theta_T\|\mathbf{Q}_N\nabla E_{12}-\nabla E_{12}\|_{\partial T_x}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(\sum_{T\in\Omega_2}h_y\left(\|\partial_x E_{12}\|_{\partial T_x}^2+\left\|\partial_y E_{12}\right\|_{\partial T_x}^2\right)+\|\partial_x E_{12}\|_T^2+\left\|\partial_y E_{12}\right\|_T^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(h_{y,N}N\varepsilon^{-2}\int_0^{x_{\frac{N}{2}-1}} e^{-\frac{2(\beta_1 x+\beta_2 y)}{\varepsilon}}dx+\sum_{T\in\Omega_2}\varepsilon^{-2}h_x h_y\left\|e^{-\frac{\beta_1 x+\beta_2 y}{\varepsilon}}\right\|_{L^\infty(T)}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon^{\frac{1}{2}}\left(\varepsilon^{-2}\left(\varepsilon\cdot N^{-2\sigma}+\varepsilon N^{-1}\cdot N^{2-2\sigma}\right)\right)^{\frac{1}{2}}\leq CN^{\frac{1}{2}-\sigma},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{T\in\Omega_0}\theta_T\|\mathbf{Q}_N\nabla E_{12}-\nabla E_{12}\|_{\partial T_x}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon\left(\sum_{T\in\Omega_0}N^{-1}\left(\|\partial_x E_{12}\|_{\partial T_x}^2+\left\|\partial_y E_{12}\right\|_{\partial T_x}^2\right)+N^{-1}h_y^{-1}\left(\|\partial_x E_{12}\|_T^2+\left\|\partial_y E_{12}\right\|_T^2\right)\right)^{\frac{1}{2}}\\
\leq & C\varepsilon\left(N^{-1}\cdot N\varepsilon^{-2}\int_{x_{N/2-1}}^1 e^{-\frac{2(\beta_1 x+\beta_2 y)}{\varepsilon}}dx+\sum_{T\in\Omega_0}\varepsilon^{-2}N^{-1}h_x\left\|e^{-\frac{\beta_1 x+\beta_2 y}{\varepsilon}}\right\|_{L^\infty(T)}^2\right)^{\frac{1}{2}}\\
\leq & C\varepsilon\left(\varepsilon^{-2}\left(\varepsilon N^{-2\sigma}+N^{-2\sigma}\right)\right)^{\frac{1}{2}}\leq CN^{-\sigma}.
\end{aligned}$$

Similar results can also be obtained for ∂T_y. Combining the above estimates, We have completed the proof of (A.3c).

Finally, for the last inequality (A.3d), note that

$$\vartheta_T = \varepsilon h_j^{-1} \leq C\varepsilon h_1^{-1} \leq CN$$

in regions Ω_1, Ω_2, Ω_{12}, and $\vartheta_T = N$ in region Ω_0. Together with (A.3b), we obtain

$$\sum_{i=x,y}\left(\sum_{T\in\mathcal{T}_N}\vartheta_T\|\mathcal{Q}_0 u - u\|^2_{\partial T_i}\right)^{\frac{1}{2}} \leq CN^{-k}.$$

At this point, all the estimates have been proven. □

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