Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 2, pp. 369-410

DOI: 10.4208/aamm.OA-2023-0250
April 2026

The Asymptotic Analysis for the Singularly Perturbed Subdiffusion Equations on Bounded Domain

Wang Kong[1,2] and Zhongyi Huang[3,*]

[1] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 211106, China*
[2] *Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing, Jiangsu 211106, China*
[3] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*

Received 21 September 2023; Accepted (in revised version) 22 April 2024

Abstract. In this paper, we study the asymptotic properties of the singularly perturbed subdiffusion equations in a bounded domain. First, we use the matched asymptotic expansion method to obtain the uniform asymptotic expansion for the solution $u(x,t)$ of the singularly perturbed subdiffusion equation. By asymptotic analysis, we can know that near the boundary configured with non-smooth boundary values, the solution $u(x,t)$ of the singularly perturbed subdiffusion equation has a boundary layer of thickness $\mathcal{O}(\varepsilon)$. By studying the asymptotic properties of the spatial partial derivatives $\partial_x u(x,t)$ and $\partial_{xx} u(x,t)$, we can know that the singularity is mainly concentrated in the boundary layers, and then the solution $u(x,t)$ changes gently outside the boundary layers. Next, we introduce a new $\mathcal{L}$1-TFPM scheme to solve the singularly perturbed subdiffusion equations numerically. Some numerical experiments can demonstrate the correctness of the asymptotic analysis results.

AMS subject classifications: 65M10, 78A48

Key words: Singularly perturbed subdiffusion equations, $\mathcal{L}$1-TFPM scheme, matched asymptotic expansion method, boundary layer.

1 Introduction

Since the 19th century, the fractional derivatives have received attention for modifying traditional physical models. In order to model the universal electromagnetic, acoustic and mechanical responses more accurately, Nigmatullin [37] uses fractional derivatives instead of time-direction derivatives to modify the classical diffusion equation. After using the fractional Fick law to replace the classical Fick's law, the subdiffusion equation,

*Corresponding author.
Emails: wkong@nuaa.edu.cn (W. Kong), zhongyih@mail.tsinghua.edu.cn (Z. Huang)

or the time-fractional diffusion equation, btained from the modeling takes the memory effect into account.

In this paper, we deal with the fractional derivative of the Caputo type. From pure mathematical viewpoint, Riemann-Liouville derivative is more welcome, but fractional differential equations in terms of the Riemann-Liouville derivatives require initial conditions expressed in terms of initial values of fractional derivatives of the unknown function [29], which are generally not physical [9]. On the contrary, initial conditions for the Caputo derivatives are expressed in terms of initial values of integer order derivatives, which have clear physical meaning [9]. For zero initial conditions the Riemann-Liouville and Caputo fractional derivatives coincide [39], and the results of this paper can be extended to Riemann-Liouville type.

In recent decades, many scholars have devoted themselves to the theoretical analysis and numerical solution of the subdiffusion equation [6,22,24,27,32,40,44,51,52]. In addition, the spatial-fractional diffusion equation has attracted the attention of many researchers [3,14–16,45,50,53]. In particular, some researchers have studied the properties and numerical solutions of steady-state spatial-fractional diffusion equations [12,13,20,31]. Besides, nonlinear time-fractional diffusion type equations are introduced to describe the anomalously subdiffusive transport behavior in heterogeneous porous materials or memory effect of certain materials [33], and then receive some attention [1,10,35,47].

When the diffusion coefficient is small, the solution of the subdiffusion equation often has a boundary layer with a sharp gradient change near the boundary, which brings great challenges to the construction of an effective numerical scheme. Scholars have paid attention to the numerical solution of the singularly perturbed subdiffusion equation with low diffusion coefficient [7,28,43,48,49]. However, there are relatively few studies on the asymptotic properties of the solutions for the singularly perturbed subdiffusion equations.

In this paper, we study the asymptotic properties for the one-dimensional singularly perturbed subdiffusion equation on a bounded domain. We consider the following initial-boundary value problem on $\Omega_B^T=[-1,1]\times(0,T]$:

$$\begin{cases} {}_0^CD_t^\alpha u(x,t)-\varepsilon^2\partial_x(a(x)\partial_x)u(x,t)=f(x,t), & (x,t)\in\Omega_B^T,\\ u(-1,t)=\phi(t), \quad u(1,t)=\psi(t), & 0\le t\le T,\\ u(x,0)=w(x), & x\in[-1,1], \end{cases} \tag{1.1}$$

where $0<\varepsilon\ll 1$ is a small parameter and the diffusion coefficient $a(x)\in C^\infty([-1,1])$ satisfies the uniform ellipticity condition, that is, there are two constants $0<a_1<a_2<+\infty$ such that

$$\bar{a}_1\le a(x)\le\bar{a}_2, \quad \forall x\in[-1,1]. \tag{1.2}$$

Here, the Caputo fractional derivative ${}_0^CD_t^\alpha w(t)$ of order $\alpha\in(0,1)$ in time direction is defined as follow

$${}_0^CD_t^\alpha w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}w'(s)ds,$$