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Nonconforming Finite Element Method for Obstacle Problem of p-Laplacian

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Abstract. We consider the nonconforming discrete Raviart-Thomas mixed finite element method (dRT-MFEM) for obstacle problems with p-Laplacian differential operator. The a posteriori and a priori error analysis were presented in a new sense of measurement. A number of experiments confirm the effective decay rates of the proposed dRT-MFEM.

AMS subject classifications: 65N12, 65N30, 65Y20

Key words: Nonconforming finite elements, p-Laplacian obstacle problem, dRT-MEFM, a posteriori error estimate.

1 Introduction

Obstacle problems are free-boundary problems for partial differential equations, wherein an inequality constrains the solution to be on one side of an identified function, the obstacle. Such problems may be formulated as variational inequalities and sometimes as constrained minimization of a functional. Numerous works [3, 8, 14, 16, 21, 23, 24] have been devoted to the mathematical analysis and numerical computation of the Obstacle problems. This paper mainly locates the obstacle problem of the p-Laplacian for steady, shallow ice sheet flows [15]. Andersson [2] proved the optimal growth result of the obstacle problem solution of p-Laplace operator and its pointwise regularity at free boundary points. Lindfors [17] studied nonlinear parabolic partial differential equations with Orlicz-type growth conditions, provided the existence of unique solutions for obstacle problems related to these equations, and proved the continuity of the solution. Figalli [11] investigated the regularity of free boundaries when the gradient vector of an obstacle is zero. Lothar Banz used a high-order coordinated finite element method for p-Laplacian

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obstacle problems [4], presented the a priori error estimation and convergence rate regarding grid size h and polynomial order q, and a general posteriori error estimation.

In this article, we focus on the nonconforming discrete Raviart-Thomas Mixed Finite Element Method (dRT-MFEM) for obstacle problems with p-Laplacian differential operator. The dRT-MFEM is a simplified MFEM with one-point numerical quadrature, it is proposed by Marini for the purpose of the cost-free approximation of Raviart-Thomas MFEM for linear problem [20]. Arbogast and Chen [1] improved the method for general variable coefficients elliptic PDEs. Carstensen and Liu [9] developed dRT-MFEM for a nonlinear Optimal design problem, and the first guaranteed energy bound and an optimal a posteriori error estimate were obtained. Recently, Liu [13, 18] generalized this method to p-Laplace equation for ($2\le p<\infty$) and $1<p<2$ respectively. The aim of this paper is to propose the nonconforming dRT-MFEM for p-Laplacian obstacle problem for $p\ge 2$ and derive the reliable a posteriori and a priori error analysis.

The rest of the paper is structured as follows. Section 2 introduces the notations of the p-Laplacian obstacle problem and its corresponding energy minimization problem. Section 3 states the dRT-MFEM and the convex property of energy density function. The a priori and a posteriori error estimates for dRT-MFEM are carried out in Section 4. Numerical experiments conclude the paper in Section 5 with empirical superiority of the proposed dRT-MFEM.

Standard notation applies throughout this paper to Lebesgue and Sobolev spaces $L^p(\Omega)$, as well as to the associated norms $\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}$, $|||\cdot|||_p := \|\nabla\cdot\|_{L^p(\Omega)}$ and $|||\cdot|||_{NC,p,\Omega} := \|\nabla_{NC}\cdot\|_{L^p(\Omega)}$ with the piecewise gradient $\nabla_{NC}\cdot|_T := \nabla(\cdot|_T)$ for all T in a regular triangulation $\mathcal{T}$ of the polygonal Lipschitz domain Ω.

2 The p-Laplace obstacle problem

Let $\Omega\subset\mathbb{R}^2$ be an open, bounded, polygonal domain with boundary $\Gamma:=\partial\Omega$. Let $2\le p<\infty$, $q=p/(p-1)$ and data $f\in L^q(\Omega)$, $\chi\in W_0^{1,p}(\Omega)$. We consider the p-Laplacian obstacle problem of finding a suitable function $u\in K$ such that

$$\begin{cases} -\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u)\ge f, & u\ge\chi, \\ (-\Delta_p u - f)(u-\chi)=0 & \text{in } \Omega, \\ u=0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where

$$K:=\left\{ v\in W_0^{1,p}(\Omega) | v\ge\chi \ \text{ a.e. in } \ \Omega \right\}$$

is the convex set of admissible functions.

Let $u=\widetilde{u}+\chi$, the corresponding mathematical model can be rewritten in finding $\widetilde{u}\in\widetilde{K}$,