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A Modified Hybrid High-Order Essentially Non-Oscillatory Compact Difference Scheme for Nonlinear Degenerate Parabolic Equations

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Abstract. In this paper, we propose a modified hybrid high-order essentially non-oscillatory compact difference scheme for both one- and two-dimensional nonlinear degenerate parabolic equations that may exist discontinuous solutions. A cell-centered compact scheme is applied to approximate the second-order derivative, where the scalar values on a cell-centered mesh are obtained by a modified hybrid essentially non-oscillatory interpolation formulation, which is constructed by coupling the compact interpolation scheme with an odd-even order weighted essentially non-oscillatory (WENO) scheme. Furthermore, a positivity-preserving limiter is designed to circumvent the numerical solutions of possible non-physical properties. The main purpose of constructing a modified hybrid cell-centered compact scheme is based on two considerations. On the one hand, it helps to avoid loss of accuracy caused by directly incorporating the WENO scheme into the cell-centered compact scheme. On the other hand, it provides a feasible way for the compact difference method to solve discontinuous problems with high-order derivatives. Here, we mainly take the numerical approximation of the second-order derivative as an example for detailed process reasoning. Through numerical tests, high-order, high resolution and essentially non-oscillatory performance of the schemes presented are verified.

AMS subject classifications: 65M06, 35K65

Key words: Compact scheme, odd-even order WENO scheme, modified hybrid interpolation scheme, nonlinear degenerate parabolic equation.

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1 Introduction

In this article, we aim to develop a hybrid high-order essentially non-oscillatory compact difference scheme for degenerate parabolic equations. Our approach is based on a cell-centered compact difference scheme and a modified hybrid essentially non-oscillatory interpolation technique, which is obtained by combining a high-order compact interpolation with an odd-even WENO interpolation scheme. We focus on addressing a class of degenerate nonlinear parabolic equations as proposed by Panov [1]. In this paper, we consider the following nonlinear degenerate parabolic equations:

$$u_t+div(\varphi(u)-\varepsilon v(u)\nabla u^m)=0, \tag{1.1}$$

satisfying the initial condition

$$u(\mathbf{x},0)=u_0(\mathbf{x}), \quad \mathbf{x}=(x_1,x_2,\cdots,x_p)\in\Omega, \tag{1.2}$$

and the suitable boundary conditions, where p represents the dimension of space, ε,m are both positive constants, the flux vector

$$\varphi(u)=(\varphi_1(u),\varphi_2(u),\cdots,\varphi_n(u)),$$

and the $v(u)$ is a function of u. However, it is important to note that the Eq. (1.1) may degenerate at some certain values of u, namely when $\nabla u^m=0$ or $v(u)=0$, or the coefficient ε is small. The degenerate parabolic equations such as Eq. (1.1) are frequently encountered in a diverse range of applications, including heat transfer processes, reservoir modeling, crystalline thermal properties, mathematical biology, collision transport models of plasma, radiation transport phenomena and flow through porous media. The Eq. (1.1) is referred to as the porous medium equation (PME) [2,3] in this case $\varepsilon v(u)\equiv 1$ with $m>1$, and $div(\varphi(u))=0$, which describes the flow of isotropic fluid in porous media. If the initial function $u_0(\mathbf{x})$ has compact support, then Eqs. (1.1) and (1.2) have at most one special solution that has compact support in space (the more details in [2,4]).

The PME exhibits a degenerate behavior at $u=0$, resulting in limited propagation velocity and a sharp front. Consequently, the Eq. (1.1) shares similarities with hyperbolic conservation laws. In conclusion, the degenerate parabolic equation (1.1) possesses characteristics akin to those observed in hyperbolic conservation laws. It is necessary to design a numerical scheme that can capture these features. At present, there are many numerical methods to approximate the solutions of degenerate parabolic equations, such as the linear approximation scheme of nonlinear Chernoff formulas [5–8], the finite volume scheme [9–11], the locally discontinuous Galerkin method [12], the relaxation scheme [13], and the kernel method [14], etc.

In this paper, we will look at the high-order compact difference method and the WENO scheme. Lele [15] proposed a cell-centered compact difference scheme (include approximation scheme with high-order derivative) based on a small stencil. Its advantage is that it can achieve high accuracy based on fewer grid points, and its resolution