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Modified One-Leg θ-Methods for Linear Neutral Pantograph Equations with Multiple Delay Terms

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Abstract. In this paper, by combining the modified one-leg θ-methods with linear interpolation, a new class of one-leg θ-methods for solving initial value problems of neutral pantograph equations with multiple delay terms is presented. Our new approach, which is based on a geometric grid, exhibits better asymptotic stability compared to traditional θ-methods. Under appropriate conditions, we prove, using the joint spectral radius method, that the proposed methods are asymptotically stable if and only if $0<\theta\leq 1$. This indicates that the new methods overcome the limitation of the traditional θ-methods, which require $\frac{1}{2}\leq\theta\leq 1$ to ensure asymptotic stability. Furthermore, several numerical experiments are provided to validate our theoretical results.

AMS subject classifications: 65L03, 65L20, 65L50

Key words: Neutral pantograph equations, multiple proportional delays, modified one-leg θ-methods, geometric grid, asymptotic stability.

1 Introduction

Consider the following initial value problems (IVPs) of linear neutral pantograph equations (NPEs):

$$\frac{d}{dt}[y(t)-N(t)y(qt)]=L(t)y(t)+M(t)y(pt)+F(t), \quad t\geq 0; \quad y(0)=y_0, \tag{1.1}$$

where $p,q\in(0,1)$ are given constants, $L(t),M(t),N(t)\in\mathbb{C}^{d\times d}$ $(d\in\mathbb{N})$ are continuous matrix-valued functions on $[0,\infty)$, whereas $F(t)\in\mathbb{C}^d$ is a continuous vector-valued function on $[0,\infty)$. These types of equations, along with their special cases, frequently arise

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in the mathematical modeling of engineering applications influenced by specific delay phenomena, such as electrodynamics, the collection of current by the pantograph of an electric locomotive, and so forth (see e.g., [1–3]).

Due to the complexity of NPEs, obtaining their analytical solutions is virtually impossible. Therefore, it is particularly important to explore numerical methods for solving NPEs and investigate the properties of numerical solutions, especially their asymptotic stability. In recent years, the numerical methods for NPEs have been extensively studied when $p=q$ and many interesting results have been reported. For the linear NPEs, Guglielmi & Zennaro [4] and Bellen, Guglielmi & Torelli [5] studied asymptotic stability of θ-methods, Katani [6] proved the convergence of multistep block methods, Zhang & Sun [7] presented the boundedness and asymptotic stability of linear and one-leg multistep methods, Koto [8], Zhao, Cao & Liu [9] and Huang & Vandewalle [10] considered the stability of Runge-Kutta methods, Huang & Gan [11] and Huang [12] attained asymptotic stability of general linear methods. For more papers relevant to NPEs, we refer the readers to [13–19] and the references therein. However, there are few references that mentioned the asymptotic stability analysis of the numerical methods for NPEs with different delay terms. To the best of our knowledge, Liu [20] proposed θ-methods with constant stepsize discretization and Zhao et al. [21] studied Runge-Kutta methods with linear interpolation for linear autonomous NPEs with different delay terms. Zhang, Xiao & Wang [22,23] derived some asymptotic estimates of the linear θ-methods with constant stepsize discretization for a class of linear nonautonomous NPEs with two delay terms.

It is noteworthy that the equations studied in [22, 23] differ slightly from (1.1). Additionally, the conditions for deriving the asymptotic estimates of linear θ-methods presented in [22, 23] are not easily verifiable. Therefore, there is a continued need to investigate the asymptotic stability of numerical methods for NPEs under more general conditions. Moreover, some researchers have studied the asymptotic stability of θ-methods for special or generalized cases of NPEs. As a result, the asymptotic stability of θ-methods with $\frac{1}{2}<\theta\leq 1$ were obtained in studies such as [4, 5, 13, 24–32], while in research such as [20, 33, 34], the authors have explored the asymptotic stability of θ-methods with $\frac{1}{2}\leq\theta\leq 1$.

The existing research results on asymptotic stability of numerical methods for NPEs primarily focus on the θ-methods with $\frac{1}{2}\leq\theta\leq 1$. We note that Ma, Gong & Liu [35] presented modified one-leg θ-methods for non-neutral autonomous pantograph equations with two delay terms and showed that these methods are asymptotically stable if $0<\theta\leq 1$. Inspired by their work, we plan to apply the modified one-leg θ-methods [35] to neutral nonautonomous problem (1.1) on a geometric grid and study the numerical asymptotic stability in this paper. When analyzing the asymptotic stability of the modified one-leg θ-methods applied to problem (1.1), the technique used in [35] is no longer applicable. Therefore, our research aims to analyze the asymptotic stability of the modified one-leg θ-methods we propose by using the joint spectral radius theory, and to prove that, under the suitable conditions, our new methods are asymptotically stable if and only if $0<\theta\leq 1$.

The outline of the present paper is as follows. In Section 2, we introduce a class of