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The Adaptive Immersed Finite Element Method for the Variational Discretization of the Elliptic Optimal Control Problem with Interface

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Abstract. In this paper, we develop the adaptive immersed interface finite element method based on the variational discretization for solving optimal control problems governed by elliptic PDEs with interface. The meshes in this method do not need to fit the interface. The state and co-state variables are discretized by piecewise linear continuous functions and the control variable is required in the variational discretization approach. A posteriori error estimates for the control, state and adjoint state are derived. New error indicators are introduced to control the error due to non-body-fitted meshes. The error estimators are implemented and tested with promising numerical results which will show the competitive behavior of the adaptive algorithm.

AMS subject classifications: 65M55, 65N30

Key words: Optimal control problem, elliptic interface problem, adaptive finite element method, variational discretization.

1 Introduction

Optimal control problems governed by elliptic PDEs with interface arise in fluid dynamics, engineering, and materials science, where the optimal control of a process is in a domain which is composed of several materials separated by surfaces. Coefficients in the elliptic PDEs may have a jump across the interface corresponding to different materials. Hence, it is a challenge to develop efficient numerical methods for such optimal control problems. Let's consider the optimal control problem:

$$\min J(y,u)=\frac{1}{2}\int_\Omega (y-y_d)^2dx+\frac{\alpha}{2}\int_\Omega u^2dx, \tag{1.1}$$

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over all $(y,u)\in H_0^1(\Omega)\times L^2(\Omega)$ subject to the elliptic interface problem

$$\begin{cases} -\nabla\cdot(A(x)\nabla y(x))=f+u & \text{in } \Omega_1\cup\Omega_2, \\ [y]_\Gamma=0, \quad [A(x)\partial_n y]_\Gamma=0, & \\ y=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and the control constraint

$$u_a\le u\le u_b. \tag{1.3}$$

Here the regularization parameter α is a fixed positive number, $\Omega\subset R^2$ is a bounded domain which is divided into subdomains Ω_1,Ω_2 by some surface $\Gamma=\Omega_1\cap\Omega_2$, see Fig. 1 for an illustration. $[v]_\Gamma$ stands for the jump of the function $v(x)$ across the interface Γ and n denotes the unit normal direction of Γ pointing to Ω_1. Across the interface Γ, the coefficient function $A(x)$ is discontinuous. For simplicity we assume in this paper $A(x)=A_i$ in Ω_i for positive constant A_i, $i=1,2$. For interface problems, it is known that optimal or nearly optimal convergence rate can be achieved if body-fitted finite element meshes are used (see e.g., [3,10]). In a body-fitted mesh, the sides intersect with the interface only through the vertices. Unfortunately, it is usually a nontrivial and time-consuming task to construct good body-fitted meshes for problems involving geometrically complicated interfaces. Therefore, numerous modified finite difference methods based only on simple Cartesian grids have been proposed in the literature. We refer to the immersed boundary method in [28], the immersed interface method in [20] and [21], the ghost fluid method in [24], and the references therein. In [22], an immersed interface finite element method was developed by local modification of finite element basis functions. However, most of the aforementioned methods assume that the underlying solutions are smooth in each sub-domain, they are not easily applied to problems involving non-smooth interfaces. [8] proposed the adaptive immersed interface finite element method for elliptic and Maxwell interface, which overcomes the shortcomings of the immersed finite element method.

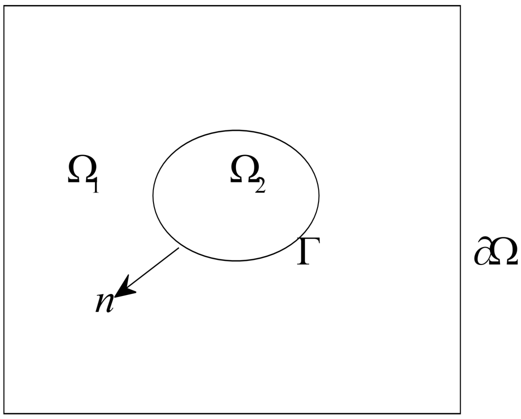

Figure 1: The geometry of the interface.