Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 2, pp. 724-755

DOI: 10.4208/aamm.OA-2023-0282
April 2026

Numerical Analysis for the Model Governing Anomalous Dynamics in Expanding Media

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Received 31 October 2023; Accepted (in revised version) 30 June 2025

Abstract. Fourier stability analysis works well and is popular for the finite difference schemes of the linear partial differential equations. In some literature, the similar idea is also used to do the convergence analysis, but they require a strong assumption on the truncation error function, which is generally impossible to achieve. Here we provide an idea of strict Fourier convergence analysis and the assumption on the truncation error function is removed. Then we apply the idea of Fourier stability and convergence analyses to the finite difference schemes of two dimensional time fractional diffusion equation with time-dependent variable coefficients, in which the time fractional derivative is discretized by the $L1$ scheme on graded meshes to overcome the weak singularity of the solution at the initial time. Finally, the numerical experiments are performed to confirm the theoretical results.

AMS subject classifications: 35R11, 65M06

Key words: Variable coefficient, $L1$ scheme, graded meshes, Fourier convergence analysis.

1 Introduction

Anomalous dynamics are ubiquitous in the nature world, especially in the complex system, including physics [1], chemistry [14], and biology [31], etc. The macroscopic equations governing anomalous dynamics are usually with nonlocal operators, e.g., fractional derivatives. Up to now, a lot of efficient numerical methods have been developed to solve fractional differential equations, including finite difference methods [5, 17, 29], finite element methods [2, 6, 13], finite volume methods [11, 19], spectral method [7], and collocation methods [16, 20] etc. The works mentioned above are all fractional differential equations with constant coefficients. Some of the models characterizing anomalous dynamics are indeed with variable coefficients [30]. There are also some discussions on the numerical methods of the equations with variable coefficients [25, 32].

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There exist many methods to analyze the stability and convergence of the numerical schemes, such as, energy method [10], analyzing matrix properties [21], etc. Among them, one of the commonly used methods is Fourier method [3, 4], especially when performing the numerical stability. Unfortunately, it seems that the existing literature using Fourier method to do convergence analysis requires a strong assumption on the truncation error function. That is, they decompose the error function as the summation of the functions with frequency $\eta_n(l)e^{ilx}$, where n denotes the time layer and $l=-\infty,\cdots,-1,0,1,\cdots,+\infty$, and require that $|\eta_n(l)|\le|\eta_1(l)|$ for all l and $n=1,2,3,\cdots,N$ [3,27]. This requirement is too strong to be satisfied. Generally, it is impossible to achieve.

To avoid this strange strong assumption, based on the superposition properties of linear equations and the idea of Fourier analysis, we provide a strict proof of Fourier convergence, which is applicable to all linear equations with constant coefficients or variable coefficients independent of space. To this end, we first illustrate our method of analysis with a basic model (2.1), and then extend the analysis to the two dimensional time fractional diffusion equation with time-dependent variable coefficients (1.1) derived in [30]. Since the model (1.1) describes the subdiffusion in expanding media, the discussion of its numerical method has clear physical motivation.

In this paper, we consider the finite difference scheme for the problem

$$\frac{\partial u(x,y,t)}{\partial t}-\frac{1}{a^2(t)}\Delta_0 D_t^{1-\alpha}u(x,y,t)=f(x,y,t), \tag{1.1}$$

where $u(x,y,t)$ is the probability density function (pdf) of finding particles at time t and position (x,y), $\alpha\in(0,1)$, Δ is the Laplace operator, and $1/a^2(t)$ is a variable diffusion coefficient. Here ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative of order $1-\alpha$ defined by [26]

$${}_0D_t^{1-\alpha}u(x,y,t)=\frac{d}{dt}\int_0^t w_\alpha(t-s)u(x,y,s)ds,$$

where $w_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(z)=\int_0^\infty s^{z-1}e^{-s}ds, \quad \mathscr{R}(z)>0.$$

According to [30], we introduce the physical background of Eq. (1.1). In general, a medium expansion brings about dramatic changes of the behavior of diffusive transport. Combining physical coordinate with comoving coordinate associated with a reference frame where the expanding medium appears to be static, based on the CTRW model [22] with long-tailed waiting time pdf, one can derive Eq. (1.1).

Various effective discretizations have been provided to approximate the time Caputo fractional derivative; see, e.g., [9,24,33]. In [18], the $L1$ method is proposed, which can obtain the optimal rate of convergence $2-\alpha$ if the solution about time is sufficiently smooth (usually it does not). It is well-known that the solution of Eq. (1.1) has a weakly singular