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Optimal Stirring Strategies for Passive Scalars in a Domain with a General Shape and No-Flux Boundary Condition

Sirui Zhu[1], Zhi Lin[1], Liang Li[1] and Lingyun Ding[2,*]

[1] *School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China*
[2] *Department of Mathematics, University of California Los Angeles, Los Angeles, CA 90095, United States*

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Abstract. Multiscale metrics such as negative Sobolev norms are effective for quantifying the degree of mixedness of a passive scalar field advected by an incompressible flow in the absence of diffusion. In this paper, we adopt a mix norm that is motivated by the Sobolev norm H^{-1} for a general domain with a no-flux boundary. We then derive an explicit expression for the optimal flow that maximizes the instantaneous decay rate of the mix norm under fixed energy and enstrophy constraints. Numerical simulations indicate that the mix norm decays exponentially or faster for various initial conditions and geometries and the rate is closely related to the smallest non-zero eigenvalue of the Laplace operator. These results generalize previous findings restricted to a periodic domain for its analytical and numerical simplicity. Additionally, we observe that periodic boundaries tend to induce a faster decay in mix norm compared to no-flux conditions under the fixed energy constraint, while the comparison is reversed for the fixed enstrophy constraint. In the special case of even initial distributions, two types of boundary conditions yield the same optimal flow and mix norm decay. These insights provide a broader framework for understanding scalar mixing across different boundary conditions, offering practical implications for optimizing flow in various physical and engineering systems.

AMS subject classifications: 76R50, 65K05, 37A25, 76F25

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1 Introduction

Mixing of scalars (mass, heat, energy, etc.) by fluid flows is a fundamental phenomenon in a wide variety of natural and industrial fluidic systems [4, 8, 36]. The mixing process

*Corresponding author.
Email: dingly@g.ucla.edu (L. Ding)

can be divided into two distinct stages [5, 7, 13]. Initially, flow advection dominates and transforms large-scale structures in the field into small-scale filaments. As the filament size decreases to the Batchelor scale, molecular diffusion takes over and homogenizes the field, smoothing out any remaining small-scale fluctuations It is noteworthy that without advection, achieving uniform homogenization solely through molecular diffusion takes a considerable amount of time even in small regions. Flow-enhanced mixing has therefore attracted significant research interests for theoretical and practical purposes [3, 9, 11, 12, 36]. It is then natural to ask how one should measure the degree of mixing and consequently, what the optimal flow is for various settings. Spatial variance is a traditional and intuitive candidate to quantify mixing. However, without diffusion the variance remains constant over time and therefore is unable to characterize mixing dynamics. By contrast, negative Sobolev norms have gained notable recognition as effective mixing measures in recent years for their guaranteed temporal decay in the ergodic sense even in the absence of diffusion [18, 22–24, 31, 32, 38, 39].

In particular, [29, 30, 38] proposed a framework to derive optimal stirring strategies in a multiple-periodic rectangle subject to different flow constraints. In this setting, the governing advection equation for the scalar field can be readily manipulated into a local-in-time optimization problem for the instantaneous decay rate of the H^{-1} scalar norm. Optimal stirring flows were analytically derived and numerically simulated afforded by simplistic Fourier representations. These idealized results provided benchmarks and key insights for practical applications [26, 35] as well as theoretical analyses [1, 14, 21, 44]. In more general scenarios, such as impermeable boundaries and non-rectangular domains, the applicability of these methods and results requires further investigation.

For this reason, in this paper we seek to extend the work in [29,30,38] to more diverse geometric shapes and to no-flux boundary conditions. While the general framework is inherited here, several theoretical and computational challenges arise and will be addressed, including a proper re-definition of the mix norm, utilizing the Leray-Helmholtz projector to enforce the no-flux condition and to derive explicit optimal stirring flows, as well as robust simulation schemes for general geometries.

The rest of the paper is organized as follows: Section 2 surveys various aspects in the formulation of the optimal stirring problem including the governing equation. In Section 3, we analyze the lower bounds of mixing rates and present the explicit formula for the optimal stirring flow. Section 4 showcases the numerical simulation of the scalar field evolution under the optimal flow for various initial conditions and domains. Additionally, we compare the optimal flow's performance under no-flux and periodic boundary conditions and discuss the impact of different types of boundary conditions on mixing efficiency. Finally, Section 5 concludes the paper by summarizing the main contributions and suggesting potential avenues for future research.

2 Preliminaries