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Taylor Remainder-Based Discontinuity Detection Method with Application to Hybrid Compact-WENO Finite Difference Scheme for Hyperbolic Conservation Laws

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Abstract. In this study, the Taylor Remainder-based Discontinuity Detection Method (TRDDM) is designed to identify shocks in numerical solutions of hyperbolic conservation laws. The TRDDM leverages the fact that Taylor remainder is minimal in smooth regions but significantly large at discontinuities. This method reduces the need for threshold parameters, making it less complex. Pre-processing the target dataset involves normalization, detection of constants, and identification of high-frequency waves. After pre-processing, Taylor remainder is calculated for the remaining dataset that contains discontinuous patterns. An appropriate partition point separates the error components, with larger errors indicating discontinuities. TRDDM analyzes local features of numerical solutions and is capable of capturing even minor discontinuities. The accuracy and efficiency of the hybrid compact-WENO scheme with TRDDM are demonstrated through classical one- and two-dimensional shocked problems.

AMS subject classifications: 65M06, 76M20

Key words: Taylor remainder, Hybrid scheme, Discontinuity detection, Hyperbolic conservation laws.

1 Introduction

The non-linear system of hyperbolic conservation laws can be written compactly as

$$\frac{\partial \mathbf{Q}}{\partial t}+\nabla\cdot\mathbf{F}(\mathbf{Q})=0, \tag{1.1}$$

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where $\mathbf{Q}$ and $\mathbf{F}(\mathbf{Q})$ represent the conservative variables and fluxes, respectively, together with appropriate initial and boundary conditions. The solutions of such non-linear systems could create both complex fine smooth and large strong gradient flow structures dynamically in space and time. The two-dimensional compressible Euler equation is:

$$\frac{\partial \mathbf{Q}}{\partial t}+\frac{\partial \mathbf{F}}{\partial x}+\frac{\partial \mathbf{G}}{\partial y}=0, \tag{1.2}$$

with

$$\begin{aligned}
&\mathbf{Q}=(\rho,\rho u,\rho v,E)^T,\\
&\mathbf{F}=(\rho u,\rho u^2+P,\rho uv,(E+P)u)^T,\\
&\mathbf{G}=(\rho v,\rho uv,\rho v^2+P,(E+P)v)^T,
\end{aligned}$$

where $\mathbf{F}$ and $\mathbf{G}$ are the fluxes in the x- and y-directions, ρ is density, E is the total energy, $(u,v)^T$ is the velocity vector, $P=(\gamma-1)(E-\frac{1}{2}\rho(u^2+v^2))$ is the pressure and $\gamma=1.4$ is the specific heat ratio of ideal gas. As classic hyperbolic conservation laws, the Euler equations often involve complex structures in solutions such as shock waves, contact discontinuities, small discontinuities, large discontinuities, and high-frequency waves in numerical simulations of fluid dynamics. Accurately capturing these complex structures in an essential oscillations-free and high-resolution way is a highly challenging problem.

A high-order and high-resolution non-linear WENO scheme [1, 3, 17, 34] can effectively address hyperbolic conservation laws with discontinuous and complex fine-scale structures. However, the WENO scheme is complex to implement, computationally expensive, and prone to excessive dissipation compared with linear schemes, such as the compact finite difference scheme [19]. A straightforward way to alleviate these difficulties is to construct a hybrid scheme, which has been widely used in the numerical simulations of fluid dynamics [4,7,11,21,25,28,31,32]. On the one hand, in the smooth regions of the flow field, the utilization of the linear, compact scheme facilitates the achievement of higher resolutions and computational efficiency. On the other hand, in the non-smooth regions, the application of the non-linear WENO scheme effectively captures high gradients and discontinuous solutions without introducing numerical oscillations. However, the performance of the hybrid scheme is heavily contingent on the accuracy, efficiency, and robustness of the shock detection method. Therefore, the key issue in the hybrid scheme is to design an accurate, efficient, and robust discontinuity detection method to divide the computational domain into smooth and non-smooth regions.

In the past 20 years, many shock detection methods have been developed under the framework of a hybrid scheme. Costa et al. [7, 8] proposed the arbitrary order multi-resolution (MR) analysis of Harten [13, 14] to identify non-smooth and smooth stencils. Li et al. [20] discussed in detail the detection effect of various low-precision discontinuity detection methods, such as the TVB method [6] and the KXRCF method [18]. The results indicated that these low-order methods can not capture discontinuities accurately in some complex structures. Don et al. [10] presented a conjugate Fourier discontinuity