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Weighted Sonine Conditions and Application

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Abstract. The Sonine kernel described by the classical Sonine condition of convolution form is an important class of kernels used in integral equations and nonlocal differential equations. This work extends this idea to introduce weighted Sonine conditions where the non-convolutional weight functions accommodate the inhomogeneity in practical applications. We characterize tight relations between classical Sonine condition and its weighted versions, which indicates that the non-degenerate weight functions may not introduce significant changes on the set of Sonine kernels. To demonstrate the application of weighted Sonine conditions, we employ them to derive equivalent but more feasible formulations of weighted integral equations and nonlocal differential equations to prove their well-posedness, and discuss possible application to corresponding partial differential equation models.

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Key words: Sonine condition, weighted Sonine condition, integral equation, nonlocal, fractional.

1 Introduction

1.1 Sonine kernels

The Sonine kernel, which is initially considered in [26], represents a particular class of kernels that have been applied in both integral equations [3,8,15,23] and nonlocal differential equations [4,16,18,28]. For $b>0$, a kernel $k(t)\in L^1(0,b)$ is called a Sonine kernel if there exists another kernel $K(t)\in L^1(0,b)$ such that the classical Sonine condition (denoted by CSC)

$$\int_0^t K(t-s)k(s)ds=1 \tag{1.1}$$

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holds for $t\in[0,b]$, where $K(t)$ is called a dual or associate kernel of $k(t)$ and is also a Sonine kernel. The Sonine kernel has favorable properties, e.g., the solutions to integral or nonlocal differential equations of convolution form with Sonine kernels could be analytically expressed with the assistance of the associate kernel, and there exist extensive investigations on the CSC [3,7,16,18,19,21].

However, as the convolution has the translation invariant feature, it is difficult to model the pointwise inhomogeneity. Instead, an additional non-convolutional weight function is usually involved to characterize such inhomogeneity. For instance, the following weighted first-kind Volterra integral equation (VIE) is widely investigated in the literature (cf. the books [1,2,5] and the references therein)

$$\int_0^t w(s,t)k(t-s)u(s)ds=f(t),\quad t\in(0,b], \tag{1.2}$$

where $w(s,t)$ is the weight function. If we intend to investigate the weighted problems such as (1.2), it is natural to propose weighted Sonine conditions (WSCs) in order to express the corresponding solutions by associate kernels as in the unweighted cases.

Conditions on w.

To investigate weighted problems, we need to specify the properties of the weight function w, which are different in different references. For instance, in Theorem 5.1.3 of the book [5], w is assumed to be continuous with $w(t,t)=1$ on $[0,b]$ and $w_2\in L^\infty(T)$, where $T:=\{(t,s):0\le s\le t\le b\}$ and $w_2(s,t):=\partial_t w(s,t)$. In Theorem 6.3.1 of the book [1], it is assumed that $w\in C^1(T)$ with $|w(t,t)|>0$ on $[0,b]$. We following these works to make the following assumptions on the weight function $w(s,t)$:

(i) $w(t,t)\neq 0$ with $\mu_*\le|w(t,t)|\le\mu^*$ for $t\in[0,b]$ for some $0<\mu_*\le\mu^*$;

(ii) $|w_2(s,t)|\le\bar{w}(t-s)$ for $0\le s\le t\le b$ for some $\bar{w}\in L^1(0,b)$.

Note that under these two conditions, w is bounded as follows

$$|w(s,t)|=\left|w(s,s)+\int_s^t w_2(s,\theta)d\theta\right|\le\mu^*+\|\bar{w}\|_{L^1(0,b)},$$

and $|w_2(s,t)|\le Q$ or $|w_2(s,t)|\le Q(t-s)^{-\sigma}$ for some $0<\sigma<1$ satisfies the condition (ii) where Q denotes a generic positive constant that may assume different values at different occurrences. Consequently, the assumptions (i)–(ii) cover a wide class of functions.

Based on the conditions on w, we introduce two WSCs, namely WSC1 and WSC2:

Definition of WSC1.

For $k\in L^1(0,b)$, there exists a $K\in L^1(0,b)$ such that

$$\int_0^t w(s,z+s)K(t-z)k(z)dz=g(s,t),\quad \forall 0\le s+t\le b, \tag{1.3}$$

for some function $g(s,t)$ satisfying the following conditions