Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 6, pp. 1625-1653

DOI: 10.4208/aamm.OA-2023-0281
December 2025

An Absolutely Stabilized Virtual Element Method for the Incompressible Stokes Equations

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Received 26 October 2023; Accepted (in revised version) 22 September 2024

Abstract. In this paper, we propose an absolutely stabilized mixed virtual element method (mixed VEM) for the incompressible Stokes equations. We employ the C^0- conforming virtual element space for the velocity and discontinuous polynomial space for the pressure. The velocity-pressure pair contains "arbitrary-order" polynomials, and stability is guaranteed by the "absolutely stabilized finite element formulation" originally introduced in [21]. We establish error estimates in the energy norm for both velocity and pressure, as well as an L^2- error estimate for the velocity. Several numerical experiments validate the theoretical analysis.

AMS subject classifications: 65N30, 76D07, 35Q35

Key words: Absolutely stabilized mixed VEM, incompressible Stokes equation, arbitrary-order polynomials.

1 Introduction

Incompressible flow phenomena governed by the Stokes and Navier-Stokes equations permeate numerous critical applications across science and engineering, including shale gas exploration, groundwater treatment, and tidal power generation. Solving incompressible flow problems accurately and efficiently remains an active and challenging research area in applied mathematics and engineering. Some key difficulties include: the saddle-point structure and coupling between pressure and velocity, satisfying the divergence-free constraint on velocity, handling complex geometries and boundary conditions, and achieving stable discretizations that converge optimally, to name a few. Driven by practical needs, various numerical methods allowing polygonal and polyhedral meshes have been developed in recent years. These include: the mimetic finite difference method (MFD) [8], the local discontinuous Galerkin method(LDG) [25], the

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hybridizable discontinuous Galerkin method (HDG) [19] the weak Galerkin finite element methods (WG) [31,32] and the the virtual element method (VEM) [9,10].

This paper focuses on VEMs which have garnered substantial research interest in recent years owing to their desirable properties like the flexibility in handling complex geometries and the ability to construct high-order accurate discretizations. VEMs build upon the ideas of MFDs within a Galerkin discretization framework. A key feature of VEMs is that the basis functions do not need to be explicitly constructed. Instead, the VEM formulation and implementation rely on appropriate projection operators, which can be computed from judiciously chosen degrees of freedom. The VEMs have developed rapidly on the basis of some pioneer works [1, 6, 7, 14, 17, 18, 20, 22], especially for the incompressible flow models [9, 10, 16, 27, 29]. Existing common mixed VEMs for incompressible flow typically approximate the velocity using virtual element spaces while the discretize the pressure in discontinuous polynomial finite element spaces. Furthermore, these spaces must be carefully selected to satisfy the discrete inf-sup condition for stability. However, satisfying the inf-sup constraint in mixed methods comes at the cost of excluding other useful element combinations that may be better suited for certain situations. Relaxing the inf-sup requirements has opened up new possibilities for mixed VEMs and combining mixed VEMs with stabilization techniques is of great significance to enable greater freedom in choosing virtual element spaces. Authors in [23] proposed a project-based stabilized VEM for the Stokes problem employing the C^0- continuous "equal-order" virtual element pairs to approximate both velocity and pressure. Authors in [24] proposed a stabilized mixed VEM for the incompressible Navier-Stokes equations by enriching the scheme with SUPG-like terms, which enabled the scheme to circumvent the inf-sup condition. Several numerical examples therein show that the scheme keeps stable with virtual element spaces containing "equal-order" polynomials, but the corresponding theoretical analysis is lacked. On the other hand, several stabilized mixed VEMs have been proposed to address challenges arising from small parameters [3,11,26].

In this work, we develop a stabilized mixed VEM inspired by the "absolutely stabilized finite element formulation" of Wang and Douglas [21] for the stationary Stokes equations. The Stokes model is one of the most representative incompressible flow problems and has been studied with various unconventional FEMs [19, 30] and stabilization techniques [4, 12]. We employ a C^0- conforming virtual element space for the velocity and discontinuous polynomials for the pressure. The velocity-pressure pair contains "arbitrary-order" polynomials, and stability is guaranteed via the nonsymmetric absolutely stabilization approach of [21]. Our method can be viewed as an extension of the stabilized method in [21] to VEM version that also circumvents inf-sup condition. A key advantage is expanding the range of stable velocity-pressure pairs beyond typical inf-sup constraints, and allowing polygonal or polyhedral meshes to enhance flexibility and adaptability to more scenarios. Error estimate in the energy norm for the velocity and pressure is established, as well as the error estimate in L^2- norm for the velocity by the dual analysis. Finally we carry out several numerical experiments to validate the theoretical analysis.