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ALM-PINN: An Adaptive Physical Informed Neural Network Optimized by Levenberg-Marquardt for Efficient Solution of Singular Perturbation Problems

Yinghao Chen[1,2], Muzhou Hou[1,*], Jingying Hu[1], Yang Xiao[3], Jinyong Ying[1] and Ting Zhang[1]

[1] *School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China*
[2] *College of Design and Engineering, National University of Singapore, Singapore 117575*
[3] *Department of Aerodynamics, College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu 210016, China*

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Abstract. This paper presents ALM-PINN, an adaptive physical informed neural network algorithm optimized by Levenberg-Marquardt. ALM-PINN is tailored to overcome challenges for solving singular perturbation problems (SPP). Traditional neural network-based solvers reframe solving differential equations task as a multi-objective optimization problem involving residual or Ritz error. However, significant disparities in the magnitudes of loss functions and their gradients frequency result in suboptimal training and convergence challenges. Addressing these issues, ALM-PINN introduces a learnable parameter for the perturbation parameter and constructs a two-terms loss function. The first loss term emphasizes approximating the governing equation, while the second term minimizes the difference between perturbation and learnable parameters. This adaptive learning strategy not only mitigates convergence issues in directly solving SPP but also alleviates the computational burden with asymptotic iteration from a large initial value. For one-dimensional tasks, ALM-PINN enhances training efficiency and reduces complexity by enforcing hard constraints on boundary conditions, streamlining the loss function sub-terms. The efficacy of ALM-PINN is evaluated on five SPPs, demonstrating its ability to capture sharp changes in physical quantities within the boundary layer, even with small perturbation coefficients. Furthermore, ALM-PINN exhibits reduced errors in both L_∞ and L_2 norms, coupled with improved convergence speed and stability.

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*Corresponding author.
Emails: chenyinghao1@csu.edu.cn, e1275353@u.nus.edu (Y. Chen), houmuzhou@sina.com (M. Hou), 8201210603@csu.edu.cn (J. Hu), xiao_yang@nuaa.edu.cn (Y. Xiao), ying48882345@gmail.com (J. Ying), tingzh_csu@foxmail.com (T. Zhang)

Key words: Singular perturbation problems, Levenberg-Marquardt, physical informed neural network, adaptive learning.

1 Introduction

Differential equations serve as fundamental tools for modeling complex systems across diverse scientific disciplines, including engineering [1–3], physics [4], economics [5–7], and biology [8–11]. Despite their power to provide insightful descriptions of natural phenomena, exact closed-form solutions to these equations are often elusive, prompting the adoption of numerical methods for practical approximations. Starting from the linear multistep methods and Runge–Kutta methods in solving ordinary differential equations (ODEs), a series of numerical analysis techniques are proposed, such as the finite difference method (FDM) [2,4], finite element method (FEM) [3,12,13]. Further enhancement strategies have been explored, such as leveraging piecewise polynomials as basis functions and designing high-order difference schemes. These improvements aim to refine the accuracy of numerical solutions.

The perturbation of physical processes associated with non-uniform transitions is elucidated through the partial dominant terms of differential operators within the governing equations [14]. These operators incorporate a diminutive parameter ϵ, manifesting an abrupt alteration in the dependent variable u, confined to a narrow layer. These intricacies are recognized as singular perturbation problems (SPP) [15]. Solving SPP accurately poses significant challenges, primarily stemming from the existence of slender regions or sharp demarcations characterized by pronounced gradients in thermal and fluid flow phenomena [16,17]. Therefore, some domain decomposition and solution decomposition techniques have been proposed to efficiently approximate the solution in these layer regions. The exploration of numerical techniques for singular perturbation equation (SPE) begins in the 1970s. While semi-robust methods like streamline-diffusion FEM [18] (SDFEM or SUPG-streamline upwind Petrov-Galerkin) perform well in regions far from the boundary layer, they often exhibit undesired spurious oscillations near the layers. Augustin et al. [19] compared various stabilized finite volume and finite element methods, highlighting that modern discretization methods such as continuous interior penalty and discontinuous Galerkin or FEM scheme with flux-correction-transport do not necessarily offer advantages. Shishkin [20] employed layer-adapted meshes for analyzing the upwind finite difference schemes. However, oscillations persist in test cases with characteristic layers and C^0 continuity by layer-adapted mesh. Furthermore, these grid refinement and region decomposition techniques require repeated solutions, leading to high computational complexity.

Neural Networks (NNs) has substantial applicability spanning diverse domains, encompassing intelligent healthcare [21–23], autonomous vehicular systems [24,25], large language models [26,27], financial investing [28–30], energy management [31–33], and