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On Stable Difference Schemes for the Solution of Soliton Type Coupled Sine-Gordon Equations in the Weak Sense

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Abstract. In the present paper numerical solution of the nonlinear coupled system of sine-Gordon equations is studied in the weak sense. A first-order accurate and two second-order accurate unconditionally stable difference schemes corresponding to the system of sine-Gordon equations are considered. Solutions of these difference schemes are presented in the space of distributions by variational methods. The fixed point theory and the finite difference method are combined in the numerical implementations, carried out in MATLAB, in order to verify the theoretical results.

AMS subject classifications: 35D30, 39A30, 65Q10, 35A15

Key words: Existence, uniqueness, weak solutions, finite difference method.

1 Introduction

Mathematical modelling of many natural phenomena, such as relativistic quantum mechanics, biomedical engineering, acoustics, and field theory problems, leads to wave equations [12–14,18,29,31] and the references given therein. Many mathematicians work on the system of wave equations in particular Klein-Gordon and sine-Gordon equations [6, 12–14, 29]. In recent years, these problems have become more interesting due to the existence of solitons. Soliton type equations appear in modelling proteins, signal processing among neurons, and deoxyribonucleic acid (DNA) [31].

Weak solutions are important in the case of low regularity of the source function, coefficients, initial, and boundary conditions. The weak solutions can be obtained even under less regularities of data in particular for systems of nonlinear equations, and for linear or semi-linear problems that don't have a mild solution. In the study of partial differential equations (PDEs), weak solutions are studied in the distribution space by using variational method, so called energy method [1–11, 22–27].

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In this study we use the theory and methodology of Roger Temam et al. [1,2,4,26,27]. This article uses second-order hyperbolic evolution equations in the following form. Let $\Omega_T=\Omega\times(0,T]$, with $T>0$, and $S=[0,T]\times\Gamma$ for $\Gamma=\partial\Omega$, $\overline{\Omega}=\Omega\cup\Gamma$, and $\Omega\subset\mathbb{R}^n$ be an open and bounded set. A widely known [6] initial/boundary-value problem is

$$\begin{cases} w_{tt}+w_t+Lw=f & \text{in } \Omega_T \\ w=0 & \text{on } S, \\ w=g, \quad w_t=h & \text{on } \in\Omega\times\{t=0\}, \end{cases} \tag{1.1}$$

with given functions $f:\Omega_T\to\mathbb{R}$, $g,h:\Omega\to\mathbb{R}$, and $w:\overline{\Omega}_T\to\mathbb{R}$ is the unknown, $w(x,t)$. Here L denotes a partial differential operator for each time t in the form

$$Lw=-\sum_{i,j=1}^{n}a^{ij}(x,t)w_{x_ix_j}+\sum_{i=1}^{n}b^i(x,t)w_{x_i}+c(x,t)w \tag{1.2}$$

for the coefficients a^{ij}, b^i, c $(i,j=1,\cdots,n)$.

In this work, the second order differential operator

$$Lw=-a(x,t)w_{xx}+b(x,t)w_x+c(x,t)w \tag{1.3}$$

for the coupled system of problem (1.1) for Ω_T and $\Omega\subset\mathbb{R}$ is considered. Here, we will assume initially the coefficients $a,b,c\in C^1\left(\overline{\Omega}_T\right)$ and $f\in L^2(\Omega_T)$, $g\in H_0^1(\Omega)$, $h\in L^2(\Omega)$.

In the present paper the weak solutions of second-order of accuracy unconditionally stable difference scheme corresponding to the nonlinear system of coupled sine-Gordon equations

$$\begin{cases} \dfrac{\partial^2u}{\partial t^2}+\alpha_{11}\dfrac{\partial u}{\partial t}+\alpha_{12}\dfrac{\partial v}{\partial t}-\beta_1\Delta u+\gamma_1\sin(\delta_{11}u+\delta_{12}v)=f & \text{in } \Omega_T, \\ \dfrac{\partial^2v}{\partial t^2}+\alpha_{21}\dfrac{\partial u}{\partial t}+\alpha_{22}\dfrac{\partial v}{\partial t}-\beta_2\Delta v+\gamma_2\sin(\delta_{21}u+\delta_{22}v)=g & \text{in } \Omega_T, \end{cases} \tag{1.4}$$

with boundary conditions

$$u=0 \quad \text{and} \quad v=0 \quad \text{on } S, \tag{1.5}$$

and initial conditions

$$u(0,x)=\varphi_1(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x)=\psi_1(x) \quad \text{in } \Omega\times\{t=0\}, \tag{1.6a}$$

$$v(0,x)=\varphi_2(x) \quad \text{and} \quad \frac{\partial v}{\partial t}(0,x)=\psi_2(x) \quad \text{in } \in\Omega\times\{t=0\}, \tag{1.6b}$$

is studied. Here, $\Omega\subset\mathbb{R}$ is a bounded open set and Δ is Laplacian. The coefficients α_{ij}, β_i, γ_i, δ_{ij}, ρ_{ij} are nonzero real numbers for $i,j=1,\cdots$. We denote

$$\begin{aligned} \tilde{f}(t,x,u,v,u_t,v_t)&=f(t,x)-\gamma_1\sin(\delta_{11}u+\delta_{12}v)-\alpha_{11}u_t-\alpha_{12}v_t, \\ \tilde{g}(t,x,u,v,u_t,v_t)&=g(t,x)-\gamma_2\sin(\delta_{21}u+\delta_{22}v)-\alpha_{21}u_t-\alpha_{22}v_t. \end{aligned}$$