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# A Ciarlet-Raviart Mixed Finite Element Method for Optimal Control Problems Governed by a Fourth Order Bi-Wave Equation

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**Abstract.** The optimal control problem governed by a stationary fourth-order bi-wave equation is considered. To tackle this problem, we propose a bilinear mixed method of the Ciarlet-Raviart type. Our method exhibits an optimal convergence rate in the $L^2$-norm and demonstrates a global supercloseness property in the $H^1$-seminorm. Moreover, through the application of an interpolation post-processing technique, the method achieves global superconvergence. We provide two numerical examples to numerically validate these theoretical properties of our proposed method.

**AMS subject classifications**: 65N30, 65N15

**Key words**: Ciarlet-Raviart scheme, mixed finite element methods, optimal control problems, fourth order bi-wave equation.

## 1 Introduction

Optimal control problems governed by partial differential equations play a pivotal role across various scientific domains, including chemical processes, fluid dynamics, medicine, and economics, as evidenced in prior works such as [3,4]. A challenge in these problems is the lack of analytical solutions or their inability to be expressed explicitly, necessitating the use of numerical methods. Among these, the finite element method [5,13] stands out as one of the most effective numerical methods and finds widespread application in solving optimal control problems. For instance, in [14], an elliptic optimal control problem

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(OCP) was addressed using a conforming finite element method, accompanied by error estimates in $L^2$-norm.

In light of optimal conditions, theoretical insights [7] have spurred extensive research into various finite element methods [8,9,19,20,32] and mixed finite element methods [10, 22, 29] for optimal control problems governed by diverse partial differential equations. For example, in [30], the convergence rate of a discretization method was enhanced by preserving the control set without explicit discretization and leveraging the relationship between the adjoint state and the control variable. Furthermore, a posteriori error estimates for finite element solutions were established for optimal control problems governed by elliptic equations [23], Stokes equations [36], and parabolic equations [21]. The exploration of nonconforming finite element methods for optimal control problems is detailed in [26–28].

The focus of the paper is on the optimal control problem of finding $(y,u)\in V\times U$, such that

$$\min_{(y,u)\in V\times U}\frac{1}{2}\int_\Omega\left[(y-y_d)^2+\alpha u^2\right] \tag{1.1}$$

subject to a stationary fourth-order bi-wave equation

$$\begin{cases}\epsilon^2\square^2 y-\Delta y=f+u & \text{in } \Omega,\\ y=\dfrac{\partial y}{\partial n}=0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^2$ is a rectangular domain for simplicity, $V=\{v\in H_0^1(\Omega):\square v\in L^2(\Omega)\}$ is chosen as the state space, $U=L^2(\Omega)$ is the control space, $\alpha(\geq C_p)$ is a positive constant, $C_p(>1)$ is the constant in Poincare inequality $\|v\|_0\leq C_p|v|_1$ for $v\in H_0^1(\Omega)$, $\epsilon\in(0,1]$ is a constant parameter, $y_d\in C^0(\Omega)$ and $f\in U$ are given functions, and the so-called bi-wave operator $\square^2$ is defined based on the operator $\square y:=\partial_{x_1}^2 y-\partial_{x_2}^2 y$ as

$$\square^2 y:=\square(\square y)=\partial_{x_1}^4 y-2\partial_{x_1}^2\partial_{x_2}^2 y+\partial_{x_2}^4 y. \tag{1.3}$$

Note that the bi-wave operator $\square^2$ is different from the biharmonic operator

$$\Delta^2 y=\partial_{x_1}^4 y+2\partial_{x_1}^2\partial_{x_2}^2 y+\partial_{x_2}^4 y. \tag{1.4}$$

The standard notations for Sobolev spaces and norms are adopted in this paper, see e.g., [5, 13]. In particular, $\|\cdot\|_p$ denotes the $H^p$-norm, $|\cdot|_p$ denotes the $H^p$-seminorm for the integer $p\geq 1$, and $(\cdot,\cdot)$ denotes the $L^2$ inner product over the domain $\Omega$.

The fourth-order bi-wave state equation (1.2) is a simplified Ginzburg-Landau-type model for $d$-wave superconductors with $y$ as the $d$-wave order parameter [12]. The state equation (1.2) alone is a linear PDE and has been studied widely. For examples, two conforming finite element methods constructed in [16], a cubic finite element method and a quartic finite element method, were suitable for different types of meshes, possess optimal order error estimates in energy norm, and achieve suboptimal error estimates