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A Discrete Collocation Technique Using the Thin Plate Splines for Solving a Certain Class of Integro-Differential Equations Arising in Biology Models

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Abstract. The primary motivation of the current article is to study an approximate algorithm to solve a type of Volterra integro-differential equations involving nonlinear terms. These integro-differential equations have been utilized to simulate several realistic modeling arising in biological sciences, for example, the growth model of the toxins' cumulative effects on a population residing in a closed system. Thin plate splines (TPSs), known as a subclass of radial basis functions (RBFs) independent of the shape parameter, create an efficient scheme to interpolate a function. Due to this desirable property, we use the discrete collocation approach with the TPS basis for estimating the solution of the mentioned integro-differential equations. Furthermore, the Gauss-Legendre integration formula is utilized to calculate the integrals emerging in the scheme. Remarkably, the algorithm of the offered approach can be comfortably performed on a PC with typical specifications derived from the simplicity of using TPSs. The researchers also investigate the convergence of the presented scheme. Some test examples including Volterra integro-differential equations are considered to make sure the veracity of the proposed scheme and the validity of the theoretical error analysis.

AMS subject classifications: 45D05, 45J05, 92-08, 92D25

Key words: Nonlinear integro-differential equation, population growth model, meshless method, discrete collocation technique, thin plate spline (TPS), error estimate.

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1 Introduction

Integro-difference equations point to any functional equations in which both integrals and derivatives of the unknown function appear together in the given equation. In the early 1900s, these classes of equations were first presented by the Italian mathematician and physicist *Vito Volterra* as models for describing population growth with an emphasis on hereditary effects [63]. It is worth noting that the nonlinear version of integro-difference equations has been applied to describe the growth of single or two species a few years later [19]. Population growth models serve as fundamental tools in understanding the dynamics of biological systems, playing a pivotal role in various fields such as ecology, epidemiology, and conservation biology. These models aim to capture the complex interplay between birth, death, immigration, and emigration rates that collectively determine changes in population size over time.

Consider a category of first-order nonlinear integro-differential equations as follows [40]:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=a(t)H(x(t))+x(t)\int_0^t K(t,s,x(s))\mathrm{d}s+f(t), \quad t,s\in[0,T], \tag{1.1}$$

with the initial condition

$$x(0)=x_0, \tag{1.2}$$

where $a(t)$ and $f(t)$ are specified functions, $x(t)$ is the unknown function to be determined, $T>0$ and x_0 are positive real constants. The given functions $H(x(t))$ and $K(t,s,x(s))$ are assumed to be nonlinear with respect to $x(t)$ and several times continuously differentiable. Also, the integral term of Eq. (1.1) represents the cumulative effect of a certain function K over the interval $[0,T]$ involving the unknown function $x(s)$. When the function H has logistic growth, this class of integro-differential equations is beneficial for modelling the population growth of unit species in a closed system. In this situation, the integral expression shows the effect of poison agglomeration on the target species and the solution $x(t)$ refers to the population of same individuals at time t [60]. These equations may be also appeared in the reformulation of some boundary problems arising from different branches of practical sciences [18,29,66].

To find analytical solutions for integro-differential equations is difficult, so many researchers pay attention to obtaining computational methods for solving various types of them. The wavelet-Galerkin [16,46], the compact finite difference [70], single-term Walsh series [57], Taylor polynomial [51], sink confluence [69], Legender polynomials [62], Taylor alignment [45], and Tau [44] schemes have been used to numerically solve Volterra integro-differential equations. Among these literature, a book written by Brunner [19] contains a large number of materials on collocation methods for the approximate solution of linear and nonlinear integro-differential equations. Among semi-analytical schemes applied for these integro-differential equations, we limit ourselves to introduce some of them such as, He's variational iteration [32], Taylor expansion [27], and Adomian decomposition [65,66] methods. Reproducing kernel algorithms have been applied for fuzzy