Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 3, pp. 947-960

DOI: 10.4208/aamm.OA-2024-0154
June 2026

The Second-Order Mass-Conservative Characteristic Finite Difference Method for Sobolev Equation with Convection Term

Kai Fu[1], Zhaohui Liu[2] and Jiansong Zhang[2,*]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China*
[2] *College of Science, China University of Petroleum, Qingdao, Shandong 266580, China*

Received 12 July 2024; Accepted (in revised version) 25 March 2025

Abstract. In this work, a new Mass-Conservative Characteristic Finite Difference Method (MCC-FDM) is developed for solving the Sobolev equation. The new method combines the method of characteristics with mass-preserving interpolation, which not only keeps mass balance but also achieves second-order accuracy in both time and space. Numerical results are presented to confirm the temporal and spatial accuracy.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Mass-conservative, finite difference, second-order accuracy in time, Sobolev equation.

1 Introduction

Sobolev equations have been widely used to model various physical processes, including the flow of fluids through fissured rock, moisture migration in soil, thermodynamics and other phenomena [1–4]. In this paper, we consider the following Sobolev equation as our model:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\nabla\cdot(c(x)u-b(x)\nabla u-a(x)\nabla u_t)=f(x,t), & x\in\Omega, \\ u(x,t)=0 & \text{on } \partial\Omega, \\ u(x,0)=u_0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^d$ $(d=1,2,3)$, $a(x)$, $b(x)$, $c(x)$, and $f(x,t)$ are known functions.

The above model problem (1.1) is evidently an advection-diffusion problem. The existence and uniqueness results for Eq. (1.1) can be found in [6, 7]. Numerically solving

*Corresponding author.
Emails: kfu@ouc.edu.cn (K. Fu), liuzhaohui2016@163.com (Z. Liu), jszhang@upc.edu.cn (J. Zhang)

Sobolev equations with convection term presents significant challenges, particularly in convection-dominated or transport cases. Conventional finite difference methods [5] or finite element methods [8–13] often produce spurious non-physical oscillations or excessive numerical dissipation. To address these issues, Douglas and Russell in [14] proposed the characteristic finite difference (C-FD) and characteristic finite element methods for convection-diffusion equation. This characteristic method effectively reduces the truncation errors in time and permits larger time steps, thereby reducing computational costs. Gao and Rui extended this technique to Sobolev equation with convection term in [15]. However, the classical characteristic method achieves only first order accuracy in time and is not mass conservative. Rui and his cooperators developed second-order characteristic methods for Darcy-Forchheimer flows [16, 17] by combining finite difference and block-centered finite difference methods, and second-order characteristic finite element schemes for convection-diffusion problems [18]. These methods, however, do not maintain mass balance. Subsequently, Rui developed a novel mass-conservative characteristic finite element scheme for convection-diffusion problems [19], and then, Zhang and his cooperators extended this technique to miscible displacement problems and wormhole propagation problems [20–24]. But this method achieves only first-order accuracy in time. For pure advection problem, Colella and Woodward introduced a piecewise parabolic method (PPM) [25], which employs parabolic interpolation at the preceding level to ensure local mass conservation for the advection equations. Fu and Liang [26–28] combined the method of characteristics with PPM to solve the advection-diffusion problems, achieving second order accuracy in time and high order accuracy, and mass conservation. These techniques were later extended to other problem with center finite difference and finite difference procedures [29–31]. To our knowledge, few methods for Sobolev equations simultaneously address diffusion terms while preserving mass conservation and achieving high-order temporal accuracy. Therefore, developing time-second-order and mass-preserving characteristic methods for solving Sobolev equations remains both important and challenging.

The main objective of this paper is to develop a new mass-conservative characteristic finite difference (MCC-FD) method for solving Sobolev equations with convection terms, employing techniques similar to those in [26–28]. The proposed method combines the method of characteristics with piecewise parabolic interpolation, enables the use of large time steps while maintaining high accuracy and preserving mass conservation. To achieve high order accuracy in time, a temporal second-order scheme is developed by averaging along the characteristics. Meantime, in order to maintain mass conservation, high order discrete scheme are presented for diffusion fluxes by approximating the cumulative mass function, ensuring the continuity of the discrete fluxes at the tracking points. Numerical experiments demonstrate that the mass-conservative characteristic finite difference method achieves second-order accuracy in both time and space while preserving mass balance.

This paper is organized as follows. In Section 2 we present the second-order conservative characteristic finite difference method for the Sobolev equation. Section 3 provides