Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **18**, No. 4, pp. 1155-1188

DOI: 10.4208/aamm.OA-2023-0307
August 2026

A Fast Spatial Two-Grid Raviart-Thomas MFE Algorithm for the Nonlinear Time Fractional Diffusion Equations on Temporal Graded Mesh

Zhichao Fang[1], Shubin Dong[1], Jie Zhao[2] and Hong Li[1,*]

[1] *School of Mathematical Sciences, Inner Mongolia University, Hohhot, Inner Mongolia 010021, China*
[2] *School of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Hohhot, Inner Mongolia 010070, China*

Received 29 November 2023; Accepted (in revised version) 21 June 2024

Abstract. In this paper, a fast two-grid algorithm is constructed for solving the nonlinear time fractional diffusion equations by using the lowest order Raviart-Thomas mixed finite element (RTMFE) space and the temporal graded mesh. In the algorithm, the Caputo time fractional derivative is discretized by the well-known $L1$ formula on the graded mesh, the spatial domain is divided into coarse and fine grids, then a fast two steps algorithm is proposed by using two-grid computing method. The existence, uniqueness and unconditional stability for the proposed algorithm on the temporal graded mesh are derived in detail. In addition, when the analytical solution satisfies different regularity assumptions, the asymptotically optimal a priori error estimates in spatial direction are obtained on both the temporal uniform and graded meshes, which show that when spatial coarse and fine grid parameters satisfy $H=O(h^{1/2})$, the fast algorithm can obtain the same accuracy as the RTMFE algorithm. Finally, two numerical examples with different regularity conditions are provided to demonstrate the theoretical results.

AMS subject classifications: 65M12, 65M15, 65M60

Key words: Two-grid method, Raviart-Thomas MFE, nonlinear time fractional diffusion equations, temporal graded mesh, error estimates.

---

1 Introduction

In this paper, we design a fast numerical algorithm for the following nonlinear time fracctional diffusion equations (TFDEs) with variable coefficient

*Corresponding author.
Email: smslh@imu.edu.cn (H. Li)

$$\begin{cases} {}_0^C D_t^\alpha u(\boldsymbol{x},t) - \operatorname{div}(\mathcal{A}(\boldsymbol{x})\nabla u(\boldsymbol{x},t)) = f(u(\boldsymbol{x},t)), & (\boldsymbol{x},t)\in\Omega\times J, \\ u(\boldsymbol{x},t)=0, & (\boldsymbol{x},t)\in\partial\Omega\times J, \\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega, \end{cases} \tag{1.1}$$

where $J=(0,T]$ with a given $T\in(0,\infty)$, $\Omega\subset\mathbb{R}^2$ is a convex and bounded domain with a polygonal boundary $\partial\Omega$, and $u_0(\boldsymbol{x})$ is the given initial data. In (1.1), the Caputo fractional derivative in temporal direction ${}_0^C D_t^\alpha u(\boldsymbol{x},t)$ is defined by

$${}_0^C D_t^\alpha u(\boldsymbol{x},t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(\boldsymbol{x},s)}{\partial s}\frac{1}{(t-s)^\alpha}ds, \quad 0<\alpha<1, \tag{1.2}$$

where $\Gamma(\cdot)$ is the common Gamma function. For the symmetric diffusion coefficient matrix $\mathcal{A}(\boldsymbol{x})$, we assume that there exist two constants $A_*,A^*>0$ such that

$$A_*\boldsymbol{z}^T\boldsymbol{z} \le \boldsymbol{z}^T\mathcal{A}(\boldsymbol{x})\boldsymbol{z} \le A^*\boldsymbol{z}^T\boldsymbol{z}, \quad \forall\boldsymbol{z}\in\mathbb{R}^2, \quad \forall\boldsymbol{x}\in\bar{\Omega}. \tag{1.3}$$

Moreover, we assume that the nonlinear source term $f\in C^2(\mathbb{R})$, and there exists a constant $L>0$ such that $|f'(u)|+|f''(u)|\le L$, which also implies f satisfies the Lipschitz condition.

It is well known that fractional diffusion equations are now important tools to describe many anomalous diffusion phenomena in science and engineering fields, especially in simulating processes with memory and genetic properties, which can achieve satisfactory results [1–3]. When problem (1.1) is a linear TFDE, that is $f(u(\boldsymbol{x},t))=\tilde{f}(\boldsymbol{x},t)$ or $\kappa u(\boldsymbol{x},t)+\tilde{f}(\boldsymbol{x},t)$ (here κ is a given constant or function), many scholars have given the existence, uniqueness, and regularity results of the analytical solution (see [4–6]), and proposed a lot of numerical methods to solve such equations (see [7–10, 13, 26]). For the nonlinear TFDE (1.1) with constant coefficient, Jin et al. [11] discussed the regularity of the analytical solution, and pointed out that if $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$ and f satisfies the Lipschitz condition, then the nonlinear TFDE has a unique solution u such that $\|\partial_t u(t)\|\le Ct^{\alpha-1}$ for $t\in(0,T]$, which means that the analytical solution u has an initial layer at $t=0$ and $\partial_t u$ blows up as $t\to 0^+$. In recent years, the initial layer problem of fractional differential equations (FDEs) has received increasing attention from many scholars, and several efficient numerical methods have been proposed to overcome it, such as corrected methods [12–15], non-uniform temporal mesh methods [16–19], and so on.

Here we focus on a very practical class of non-uniform meshes, namely graded mesh, which have been widely used to solve FDEs with initial layer [20–24]. In particular, by combining the temporal graded mesh and the classical $L1$ formula [25, 26], many scholars had done a lot of important work in solving FDEs, see [6, 27–32]. In [6], Stynes et al. proposed a finite difference (FD) method to solve a linear TFDE. In [30], Shen et al. constructed a fast FD scheme to solve a linear TFDE with a weak singularity at initial time by using the sum-of-exponential approximations. In [27], Li et al. considered a Galerkin FE method with $L1$ formula on temporal graded mesh to solve a nonlinear TFDE with non-smooth solutions by using the Newton linearized method. In [31], Zhang et al. considered a spectral method to solve a time-space fractional equation with weak initial