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A Fully Discrete Method for Space-Time Fractional Parabolic Problems

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Abstract. This paper considers a parabolic equation involving the Caputo fractional time derivative of order $\gamma\in(0,1)$ and the spectral fractional power, of order $s\in(0,1)$, of a symmetric second order elliptic operator $\mathcal{L}$. By using the Caffarelli–Silvestre extension, the original problem is converted to a quasi-stationary elliptic problem on a semi-infinite cylinder in one more spatial dimension and with a dynamic boundary condition. Based on the solution representation with the Mittag–Leffler function and the eigenpairs of the elliptic operator $\mathcal{L}$, some new regularity results are derived. For the temporal semi-discrete scheme using the Alikhanov difference approximation, stability and convergence are established. Furthermore, the first-degree tensor product finite element method is adopted for the spatial discretization to obtain a fully discrete scheme, and an error estimate is derived. Finally, an efficient algorithm based on a generalized eigenvalue problem is applied to solve the matrix system of the full discretization. Numerical examples are provided to verify the theoretical results.

AMS subject classifications: 65N30, 65M60, 65M12

Key words: Space-time fractional parabolic equation, Caffarelli–Silvestre extension, quasi-stationary elliptic problem, Alikhanov difference approximation, tensor product finite element method, generalized eigenvalue problem.

1 Introduction

Let $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ be an open polyhedron domain with boundary $\partial\Omega$. Given $T>0$, $s\in(0,1),\gamma\in(0,1)$, the forcing term f, and the initial data u_0, we consider the following space-time fractional parabolic equation:

$$\begin{cases} \partial_t^\gamma u(x',t)+\mathcal{L}^s u(x',t)=f(x',t), & (x',t)\in\Omega\times(0,T],\\ u(x',t)=0, & (x',t)\in\partial\Omega\times(0,T],\\ u(x',t)=u_0(x'), & (x',t)\in\Omega\times\{0\}. \end{cases} \tag{1.1}$$

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Here ∂_t^γ denotes the left-sided Caputo fractional derivative of order γ with respect to t, defined by

$$\partial_t^\gamma u(x',t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{1}{(t-s)^\gamma} \frac{\partial u(x',s)}{\partial s} ds.$$

The operator $\mathcal{L}^s$, $s \in (0,1)$, is the fractional power of the second order elliptic operator $\mathcal{L}$, with

$$\mathcal{L}u(x',t) := -\text{div}_{x'}(\boldsymbol{A}\nabla_{x'}u(x',t)) + cu(x',t), \tag{1.2}$$

where $\text{div}_{x'}$ and $\nabla_{x'}$ denote respectively the divergence and gradient operators with respect to x', $\boldsymbol{A} = (a_{ij}(x'))$ is a symmetric and uniformly positive definite $d\times d$ matrix function with $a_{ij} \in L^\infty(\Omega)$, and $0 \le c = c(x') \in L^\infty(\Omega)$.

The definition of $\mathcal{L}^s$ is based on the spectral theory [14]. It is standard that there exists a complete orthonormal basis $\{\phi_k \in H_0^1(\Omega) : k \in \mathbb{N}\}$ of $L^2(\Omega)$ such that

$$\mathcal{L}\phi_k = \lambda_k \phi_k, \tag{1.3}$$

where $\{\lambda_k : k \in \mathbb{N}\} \subset \mathbb{R}^+$ is a non-decreasing sequence. For any $l \in \mathbb{R}^+$, define

$$\mathbb{H}^l(\Omega) := \left\{ v = \sum_{k=1}^\infty v_k \phi_k : \ \|v\|^2_{\mathbb{H}^l(\Omega)} := \sum_{k=1}^\infty \lambda_k^l v_k^2 < \infty \right\}, \tag{1.4}$$

with $v_k = \int_\Omega v\phi_k dx'$, and denote by $\mathbb{H}^{-l}(\Omega)$ the dual space of $\mathbb{H}^l(\Omega)$. Then we define the fractional operator $\mathcal{L}^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$ by

$$\mathcal{L}^s v := \sum_{k=1}^\infty \lambda_k^s v_k \phi_k, \quad \forall v \in \mathbb{H}^s(\Omega).$$

Due to the capability of accurately describing memory and non-locality properties of various materials and processes, derivatives of fractional order are widely used in mechanics, chemistry, and engineering (cf. [3,11,12,22,24,45,54,57]). However, the nonlocal feature of the fractional operators usually causes difficulties in analysis and computation.

To overcome the the nonlocal difficulty, Caffarelli and Silvestre [9] realized the fractional Laplacian, i.e., the case of $\mathcal{L} = -\Delta$, as the Dirichlet-to-Neumann mapping for an elliptic problem in $d+1$ spatial dimensions. More precisely, they converted the fractional Laplace equation

$$(-\Delta)^s u = f, \quad x' \in \mathbb{R}^d, \tag{1.5}$$

to an extension nonuniformly elliptic problem with a Neumann boundary condition:

$$\begin{cases} \text{div}(y^\alpha \nabla \mathcal{U}(x)) = 0, & x := (x',y) \in \mathbb{R}^d \times (0,\infty), \\ \dfrac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s f, & x \in \mathbb{R}^d \times \{0\}, \end{cases} \tag{1.6}$$