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High-Order Numerical Approximation and Error Estimation for Fourth-Order Equation in Complex Cylindrical Domain

Xuenan Zhao[1], Jing An[1] and Juan Zhang[2,*]

[1] *School of Mathematical Sciences, Guizhou Normal University, Guiyang, Guizhou 550025, China*
[2] *School of Logistics, Linyi University, Linyi, Shandong 276000, China*

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Abstract. In this paper, we introduce a high-order numerical method for solving fourth-order equations in an elliptical cylindrical region. Initially, we employ an elliptical cylindrical coordinate transformation to reformulate the fourth-order equation as an equivalent second-order coupled system in the new coordinate system. To overcome the singularity introduced by coordinate transformation, an essential pole condition is derived. The weak form and its discretization are also established. Furthermore, the well-posedness of both the weak solution and its approximate solution have been theoretically investigated. By introducing novel projection operators and demonstrating their approximation properties, we prove the error estimates in conjunction with the approximation results of Fourier series. Finally, several numerical examples were provided to validate the convergence and high accuracy of our proposed schemes.

AMS subject classifications: 65M15, 65N22, 65N35

Key words: Fourth-order equation, mixed formulation, Legendre-Fourier approximation, error analysis, complex cylindrical domain.

1 Introduction

Fourth-order problems have found widespread applications in scientific research and engineering calculations [1–4]. Many complex nonlinear problems ultimately boil down to repeatedly solving a linear fourth-order problem, such as the Cahn-Hilliard equation [5, 6]. However, due to the diversity and complexity of computational domains and boundary conditions, theoretical analysis and numerical calculations have always been a concern for many scholars. Typical numerical methods include finite element

*Corresponding author.
Emails: 274200656@qq.com (X. Zhao), anjing@gznu.edu.cn (J. An), jzhang_math@hotmail.com (J. Zhang)

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methods [7–11], spectral methods [12–18], and finite difference methods [19]. Although the finite element method offers the advantage of regional flexibility, for certain high-dimensional computational domains with curved boundaries, the mesh generation can be relatively complex, and boundary approximation may introduce additional errors. On the other hand, the finite difference method is straightforward to program, and it primarily relies on low-order algorithms.

It is widely recognized that the spectral method is a high-order numerical method with spectral accuracy. It does not involve domain subdivision, so there is no additional error introduced by boundary approximations. However, it requires the computational domain to be a rectangular domain of product-type. Therefore, extending its applications for complex regions is crucial. In [20] and [21], Li and Tan et al. propose spectral methods based on dimensional reduction scheme for fourth-order problems in circular and spherical domains, respectively. In [22], An et al. present spectral approximations and error estimates for the bi-harmonic eigenvalue problem in circular, spherical, and elliptical domains. In [23], An et al. present an efficient Galerkin approximation and error analysis for the Maxwell transmission eigenvalue problem in a spherical domain. To our knowledge, there are no reports on spectral methods for fourth-order equations on elliptic cylindrical domains. The primary reason is that the elliptic cylindrical coordinate transformation not only introduces singularities and variable coefficients, but also couples the radial r and tangential θ coordinates, making it impossible to reduce the dimensionality. This poses certain challenges to both theoretical analysis and algorithm implementations.

Thus, the aim of this paper is to develop a high-order numerical method for solving fourth-order equations in an elliptical cylindrical region. Firstly, we employ an elliptical cylindrical coordinate transformation along with an auxiliary second-order elliptic equation to reformulate the fourth-order equation as an equivalent second-order coupled system in the new coordinate system. An essential pole condition is derived to overcome the singularity introduced by coordinate transformation. Then the weak form and its discretization are also established in a class of weighted Sobolev spaces and their approximation spaces. Furthermore, the well-posedness of both the weak solution and its approximate solution have been theoretically discussed. We further investigate the error estimations by novel projection operators and their approximation properties in conjunction with the approximation results of Fourier series. Finally, several numerical results were provided to validate the efficiency and accuracy of the proposed approximation schemes.

The rest of this article is organized as follows. In Section 2, we derive the equivalent reduced-order scheme and its variational forms. In Section 3, we prove the well-posedness of the solution and error estimations. In Section 4, we describe the efficient implementation of the algorithm. In Section 5, we present some numerical examples to illustrate the convergence and spectral accuracy of the algorithm. Finally, we make a short conclusion in Section 6.