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Barycentric Interpolation Collocation Methods Based on the Crank-Nicolson Scheme for Nonlinear Parabolic Optimal Control Problems

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Abstract. In this paper, two numerical schemes for nonlinear optimal control problems governed by parabolic equations are presented to deal with the challenge arising from the mutually coupled state and co-state variables in numerical simulation. Using Lagrangian multipliers, the continuous optimality system is derived, which consists of state and co-state equations, coupled with an optimality condition. To achieve higher spatial accuracy, meshless and high-precision barycentric interpolation collocation methods are applied. Fully discrete collocation approximation schemes are presented, utilizing the Crank-Nicolson scheme in time and the Newton linearization for the nonlinear term. To avoid solving the large coupled scheme directly, a classical iterative method is adopted. Furthermore, we provide consistency analyses of semi-discretized schemes in space, as well as nonlinear fully discretized schemes based on approximation properties of collocation methods. Finally, several numerical experiments are conducted to validate the efficiency of our proposed methods. Comparisons with a classical finite difference method indicate that the proposed collocation schemes offer superior accuracy while requiring fewer nodes.

AMS subject classifications: 49M25, 65M70, 65L20

Key words: Nonlinear optimal control problems, parabolic equations, Barycentric interpolation collocation, consistency analysis.

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1 Introduction

PDE-constrained optimization problems have broad applications across various research fields, including aerodynamics, geophysics, medicine, and environmental engineering. Among them, nonlinear optimal control problems governed by parabolic equations have garnered significant attention, particularly in flow control in computational fluid dynamics [1]. This trend is driven not only by its broader applications in different fields but also by the computational challenges it presents. It is widely acknowledged that the main challenge for solving the nonlinear parabolic optimal control (NPOC) problems arises from the fact that the state and co-state variables are mutually coupled and marching in opposite directions. Numerical discretization of such problems generates a large-scale sparse system of algebraic equations, as all time steps must be taken into account simultaneously [2]. In this paper, we investigate the following nonlinear parabolic optimal control problem

$$\min_{u\in U} J(y,u)=\int_0^T \left\{\frac{1}{2}\|y-\bar{y}\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}\right\}dt, \tag{1.1}$$

subject to

$$\begin{cases} y_t-\Delta y+\phi(y)=f+u & \text{in } \Omega\times(0,T],\\ y(\boldsymbol{x},t)=0 & \text{on } \partial\Omega\times(0,T],\\ y(\boldsymbol{x},0)=y_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathcal{R}^d$ $(d=1,2,3)$ represents a bounded domain with Lipschitz boundary $\partial\Omega$. $T>0$ is a finite period of time, and $\alpha>0$ is a given regularization parameter. y and u serve as the state and control variables, respectively. $\bar{y}\in L^2(0,T;L^2(\Omega))$ is the desired state, $f\in L^2(0,T;L^2(\Omega))$, and the initial condition $y_0(x)\in L^2(\Omega)$. $\phi(y)$ represents the nonlinear term. $U=L^2(0,T;L^2(\Omega))$ is a control space.

According to the optimal control theory [3], the optimal solution pair (y,u) to (1.1)-(1.2) is shown to be completely characterized by the unique solution triplet (y,p,u) to the following unconstrained first-order necessary optimality system

$$\begin{cases} y_t-\Delta y+\phi(y)=f+u & \text{in } \Omega\times(0,T],\\ y(\boldsymbol{x},t)=0 & \text{on } \partial\Omega\times(0,T],\\ y(\boldsymbol{x},0)=y_0(\boldsymbol{x}) & \text{in } \Omega,\\ -p_t-\Delta p+\phi'(y)p=y-\bar{y} & \text{in } \Omega\times(0,T],\\ p(\boldsymbol{x},t)=0 & \text{on } \partial\Omega\times(0,T],\\ p(\boldsymbol{x},T)=0 & \text{in } \Omega,\\ \alpha u=-p & \text{in } \Omega\times(0,T]. \end{cases} \tag{1.3}$$

During the past decades, some numerical algorithms have been presented to solve optimal control problems, such as finite element methods [4-6], finite difference methods [7-9], etc. Chen et al. [4] adopted the mixed finite element methods for semilinear quadratic