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New Regularization and Error Estimate for the Cauchy Problem of the Nonlinear Helmholtz-Type Equation

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Abstract. In this paper, the Cauchy problem of the nonlinear Helmholtz-type equation is discussed. This problem is well known to be severely ill-posed. Compared with linear equations, nonlinear equations are more difficult to deal with because of their lack of linearity. In order to obtain the approximate solution of this nonlinear problem, a combination of the quasi-boundary value method and the quasi-reversibility method is proposed. Using the variable separation method, the approximate solution is equivalent to solving a class of integral equations. The well-posedness of the approximation problem is proved by the Banach fixed-point theorem. Convergence analysis and error estimation are discussed. Since the convergence of error estimates cannot be obtained by the traditional a-priori bound, we introduce a new a-priori bound to obtain the convergence of error estimates, and the Hölder-type error estimate is achieved. Finally, some numerical experiments are given to corroborate the qualitative analysis and show the regularization method works well.

AMS subject classifications: 35R25, 35R30, 47A52

Key words: Cauchy problem, nonlinear Helmholtz-type equation, regularization, quasi-boundary value method, quasi-reversibility method.

1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be an open, bounded domain. The equation

$$\Delta u+k^2u=f \quad \text{in } \Omega, \tag{1.1}$$

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is called the Helmholtz-type equation in which the function $u=u(x,y)$. The constant $k\in\mathbb{C}$ is known as the wave number, which can be complex if the medium of propagation is energy absorbing. The function f is called the source term. For the case $k\in\mathbb{R}$, usually $k>0$, Eq. (1.1) is known as the Helmholtz equation, which frequently appears in many scientific and engineering applications, for example, vibration of a structure [3], the acoustics cavity problem [4], wave propagation and scattering [12, 15, 19]. For the case $\mathrm{Re}(k)=0$, i.e., $k=i\beta$, $\beta\in\mathbb{R}$, Eq. (1.1) is called the modified Helmholtz equation or Yukawa equation, which is also frequently encountered in many practical applications, such as in implicit marching schemes for the heat equation, in Debye-Huckel theory, and in the linearization of the Poisson-Boltzmann equation, which can be referred to [5] and its references. For the case $f\equiv 0$, Eq. (1.1) is known as the homogeneous Helmholtz or modified Helmholtz equation. On the contrary, if $f=f(x,y)\not\equiv 0$, Eq. (1.1) is inhomogeneous. If the source term function $f=f(x,y,u)$ depends on the unknown function u, Eq. (1.1) is known as the nonlinear Helmholtz-type equation. In the case of the wave number $k=0$ and the source term function $f\equiv 0$, Eq. (1.1) is known as the Laplace equation.

The direct problems, i.e., Dirichlet, Neumann, or mixed boundary value problems for the Helmholtz-type equation, have been studied extensively. However, in some practical problems, the boundary data on the entire boundary cannot be obtained. We can only obtain the noisy data on a part of the boundary of the concerning domain, which leads to some inverse problems. The Cauchy problem for the Helmholtz-type equation is a classical inverse problem and is severely ill-posed in the sense of Hadamard, i.e., the solution does not depend continuously on the given Cauchy data, and any small perturbation in the given data may be dramatically magnified in the solution [13, 23, 41]. Hence, it is impossible to solve the Cauchy problem of the Helmholtz-type equation by using classical numerical methods, and it requires some special techniques, for example, regularization methods.

Up to present, a huge amount of literature has been devoted to the Cauchy problem for the Helmholtz-type equation, one can refer to [2, 9–11, 20–22, 24, 25, 28–31, 33–39, 42, 45, 49–54]. At the same time, in order to obtain the approximate solution of such ill-posed problems, various regularization methods are proposed, such as the Tikhonov type regularization method [2, 9, 33, 36, 37, 50], Fourier truncated method [10, 11, 22, 36, 45, 52, 53], quasi-reversibility method [34, 35], quasi-boundary value method [31, 49, 54], filter regularization method [21, 42], spectral regularization method [50, 51], Meyer wavelet method [24], mollification method [20, 28, 30] and other methods [25, 29].

Although there are numerous works on the regularization of the Cauchy problem for the Helmholtz-type equation, most of them focus on the linear equation, including a large number of homogeneous equations [2, 9–11, 20, 24, 25, 28, 29, 33–39, 49–51, 53, 54] and a small number of inhomogeneous equations [21, 22, 30, 31, 42, 45, 52]. The literature of the Cauchy problem for the nonlinear Helmholtz-type equation setting is much more scarce, except [45]. However, many ill-posed problems with important application backgrounds, such as the Schrödinger equation for the wave function of a particle in quantum mechanics, involve nonlinear Helmholtz-type equations. This motivates us to