

Two P_4 Nonconforming Finite Elements for the Biharmonic Equation on Rectangular Meshes

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Abstract. In this paper, two new third order nonconforming finite element methods (NFEMs) on the rectangular grid are proposed for the biharmonic problem. Such finite elements are constructed by enriching 9 high-order polynomial bubbles to the P_4 polynomial space on each rectangle, so that the extra degrees of freedom can enforce the required sub-continuity of the finite element on the four edges. We prove that the methods are well-defined, have unique solution and converge at $\mathcal{O}(h^3)$ in the H^2 norm and $\mathcal{O}(h^5)$ in L^2 norm. The numerical results show that the new elements are very efficient.

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1 Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$ and outer unit normal n , the biharmonic equation is as follows

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

To introduce the variational problem, define the following Sobolev space

$$V := \left\{ v \in H^2(\Omega), v|_{\partial\Omega} = \frac{\partial v}{\partial n}|_{\partial\Omega} = 0 \right\}.$$

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By Green’s formula, the variational problem associated with (1.1) is given by: Find $u \in V$, such that

$$a(u, v) = (f, v), \quad \forall v \in V, \tag{1.2}$$

where

$$a(u, v) = \int_{\Omega} D^2 u : D^2 v dx dy,$$

$$(f, v) = \int_{\Omega} f v dx dy.$$

In the above, $D^2 v = \begin{pmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{pmatrix} v \in (L^2(\Omega))_{2 \times 2}$ is the Hessian matrix of v . Note that $a(\cdot, \cdot)$ is symmetric and positive definite on the space V and defines a norm on V as follows

$$\|\cdot\|^2 = a(\cdot, \cdot).$$

Many conforming (piecewise C^1 -polynomials) and nonconforming (non-subspace of V) finite elements are constructed for the biharmonic equation (1.1), especially in the early days of finite element research, i.e., 1960s. For example, we have the Hsieh-Clough-Tocher element (split a triangle into 3) [5,8], the Fraeijs de Veubeke-Sander element (split a quadrilateral into 4) [6,7,14], the Argyris element (C^1 - P_5 triangle) [1], the Bell element (C^1 - P_4) [2], the Bogner-Fox-Schmit rectangle (C^1 - Q_k rectangle) [3], the Powell-Sabin element (C^1 - P_2 macro-triangle) [13,15] and the Morley element (P_2 non-conforming triangle) [12, 18–20]. Some more conforming and nonconforming finite elements are constructed in the last few years, cf. [9–11, 17, 21–25].

This paper considers NFEMs (Nonconforming finite element methods) on the rectangular grid \mathbb{T}_h for solving the biharmonic problem (1.2). On \mathbb{T}_h , we suppose that any two rectangles share at most a common vertex or a common edge, and $\bigcup_{T \in \mathbb{T}_h} T = \bar{\Omega}$. Furthermore, Let h_T be the length of the long edge of the rectangle T , set $h = \max_{T \in \mathbb{T}_h} h_T$. Suppose that the grid \mathbb{T}_h satisfies the shape regularity condition, i.e., the two edges of each rectangle are of size $\mathcal{O}(h)$. Let \mathbb{E} denote the set of all edges in \mathbb{T}_h with $\mathbb{E}(\Omega)$ the set of interior edges, and $\mathbb{E}(\partial\Omega)$ the set of boundary edges, h_E the length of edge E .

In this work, we construct two high-order NFEMs. For both, the nonconforming finite element spaces include the full space of $P_4(T)$ polynomials and they satisfy the required sub-continuity constraints on the four edges of T . The two third order nonconforming finite element methods have the same degrees of freedom, but their shape function spaces are different. The first shape function space includes all P_5 polynomials and some the sixth-order polynomials, while the other one includes all P_4 polynomials and 9 bubble functions. We prove their well-definedness, uniqueness of the discrete solution, and the optimal order of convergence in the work. Further, numerical tests on these two nonconforming elements on various domains are provided, showing the high-order convergence of the methods.