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Adaptive Relaxation Strategy on Basic Iterative Methods for Solving Linear Systems with Single and Multiple Right-Hand Sides

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Abstract. Two adaptive techniques for choosing relaxation factor, namely, Minimal Residual Relaxation (MRR) and Orthogonal Projection Relaxation (OPR), on basic iterative methods for solving linear systems are proposed. Unlike classic relaxation, in which the optimal relaxation factor is generally difficult to find, in these proposed techniques, non-stationary relaxation factor based on minimal residual or orthogonal projection method is calculated adaptively in each relaxation step with acceptable cost for Jacobi, Gauss-Seidel or symmetric Gauss-Seidel iterative methods. In order to avoid the "stagnation" of the successive locally optimal relaxations, a recipe of inserting several basic iterations between every two adjacent relaxations is suggested and the resulting MRR(m)/OPR(m) strategy is more stable and efficient (here m denotes the number of basic iterations inserted). To solve linear systems with multiple right-hand sides efficiently, block-form relaxation strategies are proposed based on the MRR(m) and OPR(m). Numerical experiments show that the presented MRR(m)/OPR(m) algorithm is more robust and effective than classic relaxation methods. It is also showed that the proposed block relaxation strategies can efficiently accelerate the solution of systems with multiple right-hand sides in terms of total solution time as well as number of iterations.

AMS subject classifications: 65F10, 65B99

Key words: Adaptive relaxation, basic iterative method, minimal residual, orthogonal projection, linear system with multiple right-hand sides.

1 Introduction

In the field of computational physics, the following linear system

$$Ax=b \tag{1.1}$$

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is common, where $A \in \mathbb{R}^{(n\times n)}$ is nonsingular and usually sparse, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ stand for the right-hand side vector and the vector of unknowns, respectively.

In practice, the solution strategies for solving linear systems with smaller n are based on direct methods, because these direct methods are robust and the characteristics of solution are predictable. However, in direct methods, the sparsity of A is usually not easy to keep after matrix factorization and thus excessive storage is required. This makes the direct approaches have poor scaling with system size. Increasing needs for solving large sparse linear systems triggers a shift toward iterative techniques [1]. Basic iterative methods such as Jacobi and Gauss-Seidel iterative methods, suffer from significant slow convergence for "low frequency error component" [2]. This makes Krylov subspace approaches such as the Conjugate Gradient (CG) method [3] and the Generalized Minimal Residual (GMRES) method [4] more attractive.

Research on acceleration strategies for basic iterative methods is motivated by their conciseness and parallelizability as well as their characteristics of smoothing high frequency error for multigrid method [5, 6]. Although basic iterative methods are hardly used as standalone solvers due to their slow convergence rate, they are important parts of high performance parallel or multigrid algorithm for larger systems. In addition, the performance of basic iterative methods is also important when they are used as preconditioners for Krylov subspace methods [7].

Among the acceleration strategies, Chebyshev acceleration [5,8] and relaxation acceleration [5] are common techniques. Chebyshev acceleration is essentially a kind of polynomial acceleration method and the optimal Chebyshev acceleration requires knowledge of eigenvalues of A. Classic relaxation (or extrapolation) strategy [9–11] can accelerate basic iterative methods by introducing the so-called relaxation factor. When applied to Jacobi iterative method, the classic relaxation method yields to Weighted Jacobi method [1] or JOR (Jacobi Over-Relaxation) method [11]. The Successive Over Relaxation (SOR) is another kind of relaxation acceleration strategy for Gauss-Seidel iterations.

The performance of relaxation method is dependent on and sensitive to the choice of relaxation factor. In practice, the relaxation factor can be specified empirically or calculated through some techniques. Calculating the optimal value of relaxation factor is not easy and there is a vast number of works on estimating the factor in the literature. Two categories of the techniques are classified here: the analysis approach and the numerical approach.

The analysis approaches try to give reference formulas analytically of the optimal relaxation factor by utilizing properties of the iteration matrix, for example, the formulas for Weighted-Jacobi as well as Gauss-Seidel relaxation methods [12]. These formulas are related to eigenvalues of the iteration matrix which are difficult or even impossible to obtain, and suitable only for specific situation. The implementation of the formulas may vary with different problems or techniques. For finite difference problem to the Poisson equation, Yang and Gobbert [13] provided an explicit SOR relaxation factor formula form for different space dimensions; in [14], another mathematical analysis was conducted to derive the eigenvalues of the iteration matrix. Implementation of the SOR relaxation