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Error Estimates for Finite Element Approximation to Elliptic Optimal Control Problems with Boundary Observations in $H^{-\frac{1}{2}}(\Gamma)$

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Abstract. In this paper we consider the finite element approximation to an elliptic optimal control problem with boundary observations in $H^{-\frac{1}{2}}(\Gamma)$. This problem is motivated by certain applications in optimal control of semiconductor devices where the discrepancy of the current density on the boundary to the target one is the objective to be minimized. The observation in $H^{-\frac{1}{2}}(\Gamma)$ is realized through a Neumann-to-Dirichlet mapping which facilitates the theoretical and numerical analysis. A priori error estimate for the optimal control is derived based on the variational control discretization, whereas the state and adjoint state variables are approximated by piecewise linear and continuous finite elements. Second order convergence rate for the optimal control is theoretically proved and verified by numerical results.

AMS subject classifications: 49M25, 49M41, 49K20, 65N30

Key words: Optimal control problems, boundary observations, finite element methods, error estimate.

1 Introduction

In this paper, we consider the following elliptic distributed optimal control problem

$$\min_{u\in U_{ad}} J(y,u)=\frac{1}{2}\|\partial_n y-y_N\|^2_{-\frac{1}{2},\Gamma}+\frac{\alpha}{2}\|u\|^2_{0,\Omega_U} \tag{1.1}$$

subject to

$$\begin{cases} -\Delta y=\chi_{\Omega_U}u & \text{in } \Omega, \\ y=0 & \text{on } \Gamma. \end{cases} \tag{1.2}$$

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The main feature of this problem is that the cost functional involves boundary observations in $H^{-\frac{1}{2}}(\Gamma)$-norm. The primary goal of this problem is to find the optimal control u such that the discrepancy between the outward normal derivative $\partial_n y$ and the boundary observations $y_N \in H^{-\frac{1}{2}}(\Gamma)$ is minimized.

Here we assume that $\Omega \subseteq \mathbb{R}^2$ is an open bounded polygonal domain with a boundary $\Gamma := \partial\Omega$. $\Omega_U \subset \Omega$ is the control domain and χ_{Ω_U} is the characteristic function of Ω_U. $\alpha > 0$ is a fixed regularization parameter, n is the unit outward normal vector on the boundary. We denote the admissible set of controls by

$$U_{ad} = \{u \in L^2(\Omega_U) : a \leq u \leq b \ \text{ a.e. in } \ \Omega_U\}, \tag{1.3}$$

where a and b are given real numbers.

PDE-constrained optimal control problems find many applications in science and engineering. One of the typical applications is the glass cooling (cf. [14, Chap. 4]). During cooling procedure the large temperature gradients have to be avoided because they usually lead to thermal stress in the material, so that the cracks may appear in the resulting product which affects the optical quality greatly. Hence, the process has to be manipulated in such a way that temperature gradients are sufficiently small, which can be realized by controlling the magnitude of the heat source and imposing additional state constraints. Another typical application of optimal controls appears in the optimal design of semiconductor devices (cf. [14, Chapter 4] and [13]). The main purpose of the control process is to gain an amplified current at the working point only by a slight change of the doping profile, where the doping profile serves as the control. This application also motivates this work because the discrepancy between the outward normal derivative $\partial_n y$ on the boundary and the target y_N is the quantity of interest to be minimized. A similar problem as in (1.1)-(1.2) has been studied in [20] where the observations are in $L^2(\Gamma)$. This setting requires some additional regularity for the solution of the state equation, which may be, however, not fulfilled by the governing drift-diffusion equations in semiconductor modeling. This work can be viewed as a preparation for studying optimal controls of semiconductor devices.

Numerical method for PDE-constrained optimal control problems is a hot research topic in the past two decades. Among different approaches, finite element method is definitely one of the most popular methods. Great achievements have been made to solve different kinds of optimal control problems, including the distributed control (cf. [7, 12]) and the boundary control (cf. [1, 3, 4, 9]), as well as control problems with different forms of observations (cf. [5, 20]). Besides the standard finite element method, different types of finite element methods have also been used to solve PDE-constrained optimal control problems, including the mixed finite element method (cf. [6,11]), the hybridizable discontinuous Galerkin method (cf. [15]), the local discontinuous Galerkin method ([21]) and the virtual element method (see [18]), to name a few.

In this paper we intend to derive a priori error estimate for the finite element approximation to the control problem (1.1)-(1.2). To obtain the first order optimality condition of the control problem, we need to give a suitable definition for the $H^{-\frac{1}{2}}(\Gamma)$-norm