Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **16**, No. 6, pp. 1328-1357

DOI: 10.4208/aamm.OA-2023-0053
December 2024

A New Reduced Basis Method for Parabolic Equations Based on Single-Eigenvalue Acceleration

Qijia Zhai[1], Qingguo Hong[2] and Xiaoping Xie[1,*]

[1] *School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, China*
[2] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409 USA*

Received 15 February 2023; Accepted (in revised version) 4 December 2023

Abstract. In this paper, we develop a new reduced basis (RB) method, named as Single Eigenvalue Acceleration Method (SEAM), for second order parabolic equations with homogeneous Dirichlet boundary conditions. The high-fidelity numerical method adopts the backward Euler scheme and conforming simplicial finite elements for the temporal and spatial discretizations, respectively. Under the assumption that the time step size is sufficiently small and time steps are not very large, we show that the singular value distribution of the high-fidelity solution matrix U is close to that of a rank one matrix. We select the eigenfunction associated to the principal eigenvalue of the matrix $U^\top U$ as the basis of Proper Orthogonal Decomposition (POD) method so as to obtain SEAM and a parallel SEAM. Numerical experiments confirm the efficiency of the new method.

AMS subject classifications: 65M22, 65M60, 65Y10

Key words: Reduced basis method, proper orthogonal decomposition, singular value, second order parabolic equation.

1 Introduction

The Reduced Basis (RB) method is a type of model order reduction approach for numerical approximation of problems involving repeated solution of differential equations generated in engineering and applied sciences. It was first proposed in [2] for the analysis of nonlinear structures in the 1970s, and later extended to many other problems such as partial differential equations (PDEs) with multiple parameters or different initial conditions [4, 9–12, 39, 41, 43], PDE-constrained parametric optimization and control problems [16–18,36], and inverse problems [24,33]. It should be mentioned that the work

*Corresponding author.
Email: xpxie@scu.edu.cn (X. Xie)

in [40, 50] has led to a decisive improvement in the computational aspects of RB methods, owing to an efficient criterion for the selection of the basis functions, a systematic splitting of the computational procedure into an offline (parameter-independent) phase and an online (parameter-dependent) phase, and the use of posteriori error estimates that guarantee certified numerical solutions for the reduced problems. The RB methods that include the offline and online phases as their essential constituents have become the most widely used ones.

The Proper Orthogonal Decomposition (POD) method, combined with the Galerkin projection method, is a typical RB method. It uses a set of orthonormal bases, which can represent the known data in the sense of least squares, to linearly approximate the target variables so as to obtain a low-dimensional approximate model. Since it is optimal in the least square sense, the POD method has the property of completely relying, without making any a priori assumption, on the data.

The POD method, whose predecessor was initially presented by K.Pearson [38] in 1901 as an eigenvector analysis method and was originally conceived in the framework of continuous second-order processes by Berkooz [5], has been widely used in many fields with different appellations. In the singular value analysis and sample identification, the method is called the Karhumen-Loeve expansion [21]. In statistics it is named as the principal component analysis (PCA) [19]. In geophysical fluid dynamics and meteorological sciences, it is called the empirical orthogonal function method (EOF) [35,44]. We can also see the widespread applications of POD method in fluid dynamics and viscous structures [3,8,15,25,37,42,45–47], optimal fluid control problems [20,32], numerical analysis of PDEs [1,6,22,23,26–31] and machine learning [7,13,34,48,49].

In this paper, we develop a new RB/POD method, named as Single Eigenvalue Acceleration Method (SEAM), for a full discretization, using backward Euler scheme for temporal discretization and continuous simplicial finite elements for spatial discretization, of second order parabolic equations with homogeneous Dirichlet boundary conditions. The idea of SEAM is inspired by a POD numerical experiment for a one-dimensional heat conduction problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}, & x\in(0,1), \quad t\in(0,0.1], \\ u(x,0)=\sin(4\pi x), & x\in(0,1), \\ u(0,t)=u(1,t)=0, & t\in(0,0.1]. \end{cases} \tag{1.1}$$

When we focus on the numerical solution matrix U of (1.1), where each column consists of the value of the numerical solution at a node, and the number of columns is the same as the number of time steps (We divide $(0,T]$ into 1000 equal parts and D into 99 equal parts to get the high-fidelity numerical solution; see Fig. 1), we observe something interesting: the principal singular value is much larger than other singular values of the numerical solution matrix! Here we recall that the singular values of U are the arithmetic square roots of the eigenvalues of $U^\top U$. Notice that when we use the POD method the choice of