Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **16**, No. 6, pp. 1451-1473

DOI: 10.4208/aamm.OA-2022-0097
December 2024

A Second-Order Alikhanov Type Implicit Scheme for the Time-Space Fractional Ginzburg-Landau Equation

Yufang Gao[1], Shengxiang Chang[2] and Changna Lu[2,*]

[1] *School of Ecology and Applied Meteorology, Nanjing University of Information Science and Technology, Nanjing, Jiangsu 210044, China*
[2] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, Jiangsu 210044, China*

Received 4 April 2022; Accepted (in revised version) 6 June 2024

Abstract. In this paper, we consider an implicit method for solving the nonlinear time-space fractional Ginzburg-Landau equation. The scheme is based on the $L2$-1_σ formula to approximate the Caputo fractional derivative and the weighted and shifted Grünwald difference method to approximate the Riesz space fractional derivative. In order to overcome the non-local property of Riesz space fractional derivatives and the historical dependence brought by Caputo time fractional derivatives, this paper introduces the fractional Sobolev norm and the fractional Sobolev inequality. It is proved in detail that the difference scheme is stable and uniquely solvable by the discrete energy method. In particular, the difference scheme is unconditionally stable when $\gamma \leq 0$, where γ is a coefficient of the equation. Moreover, the scheme is shown to be convergent in l_h^2 norm at the optimal order of $\mathcal{O}(\tau^2+h^2)$ with time step τ and mesh size h. Finally, we provide a linearized iterative algorithm, and the numerical results are presented to verify the accuracy and efficiency of the proposed scheme.

AMS subject classifications: 35Q56, 65M06, 65M12

Key words: Time-space fractional Ginzburg-Landau equation, Caputo fractional derivative, Riesz fractional derivative, $L2-1_\sigma$ formula, convergence.

1 Introduction

The classical complex Ginzburg-Landau (GL) equation is one of the physics community's most studied nonlinear equations. It was proposed in the low-temperature superconducting model by physicists Ginzburg and Landau [12]. It describes a variety of phenomena from nonlinear waves to second-order phase transitions, from superconductivity, superfluidity, and Bose-Einstein condensation to liquid crystals and strings in field

*Corresponding author.
Emails: gaoyf@nuist.edu.cn (Y. Gao), changsx7818@gmail.com (S. Chang), luchangna@nuist.edu.cn (C. Lu)

theory [4]. The global existence, regularity, asymptotic stability, well-posedness, and the long-time behavior of the exact solution were discussed in [6–8].

In recent years, fractional differential operators have become important tools for describing various complex mechanics and physical behaviors [24]. For example, fractional differential equations are widely used in anomalous diffusion [22], fluid mechanics, electromagnetic waves, quantum mechanics [16, 17]. The space fractional Ginzburg-Landau (SFGL) equation was proposed from the variational Euler-Lagrange equation for fractal media by Tarasov and Zaslavsky [25, 26].

Since the analytical solution of fractional differential equations is challenging to obtain, it is especially critical to construct effective numerical methods. These include the finite difference method [5, 10, 31, 38], finite element method [18], spectral method [20]. For the nonlinear SFGL equation, there have been some results. Mvogo [23] proposed a semi-implicit Riesz fractional finite difference scheme and proved the order of convergence is $\mathcal{O}(\tau+h^2)$. Wang and Huang [28] proposed an implicit midpoint difference scheme for the nonlinear complex SFGL equation involving fractional Laplacian. Later, they proposed a high order implicit-explicit difference scheme [29] with second-order in time and fourth-order in space. Li and Huang [18] constructed a Galerkin finite element method for the nonlinear SFGL equation. Hao and Sun [14] adopted the Crank-Nicolson/leapfrog method in temporal discretization and used the fourth-order quasi-compact format in spatial discretization to construct a linearized semi-implicit scheme. He and Pan [15] developed a linearized implicit finite difference scheme and proved the stability and second-order accuracy in both time and space variables. Zeng and Xiao et al. [34] discussed the error estimate of the Fourier pseudo-spectral method for space fractional Ginzburg-Landau equation. Zhang and Lin et al. [36] developed two types of linearized ADI schemes for two-dimensional space fractional complex GL equation and proved the unique solvability and stability. Fei and Huang et al. [11] constructed a linearized Galerkin-Legendre spectral method for the one-dimensional nonlinear SFGL and proved the convergent in the maximum norm. Wang [30] proposed a fast spectral-Galerkin method for the nonlinear SFGL equation with second-order accuracy in time and algebraical accuracy in space. As a particular form of fractional Ginzburg-Landau equation, the fractional Schrödinger equation was studied extensively in literature [9, 10, 19, 27, 35, 37].

However, there is little attention to the numerical solutions of the nonlinear fractional GL equation with fractional derivatives both in time and space. Zaky and Hendy et al. [32] proposed a numerical algorithm for the time-space fractional Ginzburg-Landau equation by Galerkin spectral scheme. Recently, they [33] proposed a high order Alikhanov Legendre-Galerkin spectral method for the nonlinear coupled time-space fractional Ginzburg-Landau equation. The considerable challenges such as historically in time and nonlocality in space and the nonlinear term of the Ginzburg-Landau equation motivate us to consider this case of study.

The Ginzburg-Landau type equation, proposed in many mechanics, physics, and chemistry problems, is an essential nonlinear evolution equation with various forms. In