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Superconvergence Error Estimate of the Bilinear-Constant Scheme for the Stokes Equations with Damping

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Abstract. In this paper, the superconvergence error estimate of a low-order conforming mixed finite element scheme, which is called bilinear-constant scheme, for the Stokes equations with damping is established. In terms of the integral identity technique and dealing with the damping term carefully, the superclose estimates between the interpolation of the exact solution and the finite element solution for the velocity in H^1-norm and the pressure in L^2-norm are first derived. Then, the global superconvergence results for the velocity in H^1-norm and the pressure in L^2-norm are derived by a simple postprocessing technique with an economical workload. Finally, some numerical results are presented to demonstrate the correctness of the theoretical analysis.

AMS subject classifications: 65M15, 65M60, 65N12

Key words: Stokes equations with damping, bilinear-constant scheme, superclose and superconvergence estimates.

1 Introduction

This paper focus on the superconvergence error estimate with a low-order conforming mixed finite element method for the following stationary Stokes equations with damping:

$$\begin{cases} -\nu\Delta \boldsymbol{u}+\alpha|\boldsymbol{u}|^{r-2}\boldsymbol{u}+\nabla p=\boldsymbol{f}, & \boldsymbol{x}\in\Omega, \\ \nabla\cdot\boldsymbol{u}=0, & \boldsymbol{x}\in\Omega, \\ \boldsymbol{u}=0, & \boldsymbol{x}\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a rectangular domain with boundary $\partial\Omega$ and $\boldsymbol{x}=(x,y)$. $\boldsymbol{u}=(u_1,u_2)$ is the fluid velocity, p is the pressure and $\boldsymbol{f}$ is a given external force, respectively. The parameter ν is the viscosity and $|\cdot|$ denotes the Euclidean norm. Moreover, the Forchheimer (or

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damping) term $\alpha|\boldsymbol{u}|^{r-2}\boldsymbol{u}$ with coefficient $\alpha>0$ and $2\le r<\infty$ comes from Forchheimer's law [1], which describes various physical situations such as porous media flow, drag, or friction effects and some dissipative mechanisms [2,3].

The mathematical analysis and numerical approximation of (1.1) have attracted considerable interests. The existence and uniqueness of the weak solution and the discrete solution were proven in [4], and the corresponding optimal error estimates were obtained for conforming mixed finite element methods. In [5], a local projection stablization method was used to overcome the lack of the inf-sup condition with low order mixed finite element spaces, and optimal error estimates for the velocity and the pressure were derived. The mark and cell method was applied to discretize (1.1) on the non-uniform grids, and the stability and the error estimates were studied in [6]. In addition, in [7], a class of conforming finite element methods were employed to (1.1) and the superclose and superconvergence for the velocity in H^1-norm and the pressure in L^2-norm were obtained with some restrictions on ν due to estimate the damping term roughly. Meanwhile, the degrees of freedom is larger than the bilinear-constant scheme presented in this paper. Moreover, we refer the readers to [8–10] for the Navier-Stokes equations with damping.

On the other hand, it is well known that superconvergence is an efficient procedure for improving the accuracy of the approximation solutions in numerical analysis [11,12]. We refer the readers to [13–23] and the references cited therein for the superconvergence analysis for different problems. Especially, the supeconvergence analysis was investigated for Stokes and Navier-Stokes equations by low order nonconforming finite element method in [22] and [23], respectively. In addition, the superconvergence analysis was studied by a stable conforming bilinear-constant scheme for Stokes equations over a uniform rectangular mesh in [24]. The bilinear-constant scheme was also applied to optimal control problems governed by Stokes equations and the global superconvergence analysis for the finite element approximation was discussed in [25]. Compared with Navier-Stokes equations, Stokes equations with damping are more complicated for the nonlinear damping term $\alpha|\boldsymbol{u}|^{r-2}\boldsymbol{u}$, while Navier-Stokes equations have a linear damping term.

This paper studies the superconvergence property by the bilinear-constant scheme used in [24,25] for the Stokes equations with damping over a uniform rectangular mesh. The superclose error estimates between the interpolation of the exact solution and the finite element solution are first derived by the integral identity technique [11,12,24,25] and dealing with the damping term carefully. Then, the global superconvergence results are obtained by a postprocessing technique. Finally, some numerical results are presented to verify the theoretical findings.

The outline of this paper is as follows. In Section 2, we introduce some preliminaries and recall some lemmas, which are necessary in the following error estimates. In Section 3, we present the detailed superclose and superconvergence error analysis for problem (1.1). In Section 4, we provide some numerical results to verify the theoretical analysis. Some conclusions are given in the final section.