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An Improved WENO-Z+ Scheme

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Abstract. The WENO-Z+ scheme [F. Acker, R. B. de R. Borges, and B. Costa, An improved WENO-Z scheme, J. Comput. Phys., 313 (2016), pp. 726–753] with two different versions further raised the nonlinear weights with respect to the nonsmooth or less-smooth substencils, by introducing an additional term into the weights formula of the well-validated WENO-Z scheme. These two WENO-Z+ schemes both produce less dissipative solutions than WENO-JS and WENO-Z. However, the recommended one which achieves superior resolutions in the high-frequency-wave regions fails to recover the designed order of accuracy where there exists a critical point, while the other one which obtains the designed order of accuracy at or near critical points is unstable near discontinuities. In the present study, we find that the WENO-Z+ schemes over-amplify the contributions from less-smooth substencils through their additional terms, and hence their improvements of both stability and resolution have been greatly hindered. Then, we develop improved WENO-Z+ schemes by making a set of modifications to the additional terms to avoid the over-amplification of the contributions from less-smooth substencils. The proposed schemes, denoted as WENO-IZ+, maintain the same convergence properties as the corresponding WENO-Z+ schemes. Numerical examples confirm that the new schemes are much more stable near discontinuities and far less dissipative in the region with high-frequency waves than the WENO-Z+ schemes. In addition, improved results have been obtained for one-dimensional linear advection problems, especially over long output times. The excellent performance of the new schemes is also demonstrated in the simulations of 1D and 2D Euler equation test cases.

AMS subject classifications: 65M06, 65M12

Key words: WENO scheme, high resolution, over-amplification, hyperbolic problems.

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1 Introduction

The first WENO (weighted essentially non-oscillatory) scheme was developed by Liu et al. [1] as an improvement to the ENO (essentially non-oscillatory) schemes [2–5]. Later, it was developed to recover the designed order of accuracy [6] resulting in the well-known WENO-JS scheme. Since this classical WENO-JS [6] scheme was proposed, the WENO schemes have been widely investigated [7–21, 48–51]. The WENO schemes owe their success to the ENO property near discontinuities and high-order accuracy in smooth regions.

It was pointed out by Henrick, Aslam and Powers [7] that WENO-JS may fail to obtain the designed order of accuracy at a point where the first derivative is zero while the second derivative is non-zero. To amend this drawback, they [7] devised the famous WENO-M scheme by constructing a mapping function to correct the classical WENO-JS weights. After that, a series of new mapping functions were designed leading to various versions of mapped WENO schemes [13–16,18–22]. As complicated mapping procedures are often required, the extra computational costs of most mapped WENO schemes are expensive.

Inspired by the work of Henrick et al. [7], Borges et al. [8] innovatively developed a very different way to determine the weights by incorporating the higher-order information about the numerical solution in the form of a global smoothness indicator (abbreviated to GSI). The new weights can satisfy the sufficient conditions for optimal convergence order, and thus, the new scheme, dubbed WENO-Z, recovers optimal convergence order even near critical points. Moreover, WENO-Z gains better results than WENO-M due to its lower dissipation while only consuming almost the same computational cost as that of the WENO-JS scheme. By designing different smoothness indicators or GSIs, a series of WENO-Z-type schemes [9–12, 23–26] were devised to further extend WENO-Z.

Compared to WENO-JS, WENO-M and WENO-Z both produce much less dissipation at or near critical points. And also, in high-frequency-wave regions, they both obtain greatly improved resolutions, especially for the standard shock-density wave interaction problems [27–30]. Since WENO-M and WENO-Z both show better convergence properties near the peak of the waves, the improved resolution of high-frequency waves was attributed to better accuracy at critical points. Until recently, it was revealed by Acker et al. [9] that assigning larger values to the nonlinear weights with respect to the nonsmooth or less-smooth substencils is the most relevant cause. Following this thought, Acker et al. [9] developed the improved WENO-Z scheme, dubbed WENO-Z+, by introducing an additional term into the weights formula of WENO-Z to further raise the nonlinear weights with respect to nonsmooth or less-smooth substencils. Two versions of WENO-Z+ were reported and both of them gained greatly better solutions compared to WENO-JS and WENO-Z on solving problems involving fine smooth structures and shocks [9]. However, increasing the nonlinear weights with respect to nonsmooth or less-smooth substencils also raises the potential risk of numerical instability. Therefore, the recommended version of WENO-Z+ enhances its stability at the price of violating the