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A Nonconvex and Nonsmooth Model for Deblurring Images under Cauchy Noise

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Abstract. In this paper, we propose a nonconvex and nonsmooth model for image restoration with Cauchy noise removal. The regularization term is composed of a nonconvex Lipschitz potential function with first-order derivatives of an image, while the data fidelity term is introduced based on the maximum a posteriori estimator to Cauchy distribution. To handle the nonconvexity of regularizers, we adopt the proximal linearization technique to convert the original nonconvex model to a series of convex models, which can be easily implemented by alternating direction method with multipliers. Based on the Kurdyka-Łojasiewicz property, we can verify the global convergence of the proposed algorithm by using an abstract convergence framework. Numerical experiments and comparisons indicate that our method obtains good restorations and is effective for better preserving edges.

AMS subject classifications: 49K30, 49N45, 49N60, 90C26, 94A08, 94A12

Key words: Nonconvex and nonsmooth, Kurdyka-Łojasiewicz property, Cauchy noise, image restoration.

1 Introduction

In image processing, images are unavoidable to be degraded by blur and noise because of the limitation of imaging devices. Except for the common Gaussian noise [11, 29, 35, 37], Cauchy noise is another important type of additive noise appeared in wireless communication systems, biomedical images and synthetic aperture radar images [21,30,42]. In this paper, we focus on designing a nonconvex nonsmooth restoration model with globally convergent algorithm for deblurring images under Cauchy noise.

Without loss of generality, we rearrange an $n\times n$ image U into a vector $u\in\mathbb{R}^m$. Assume that $f\in\mathbb{R}^m$ is an observed image degraded from the true image $u\in\mathbb{R}^m$ with blur and

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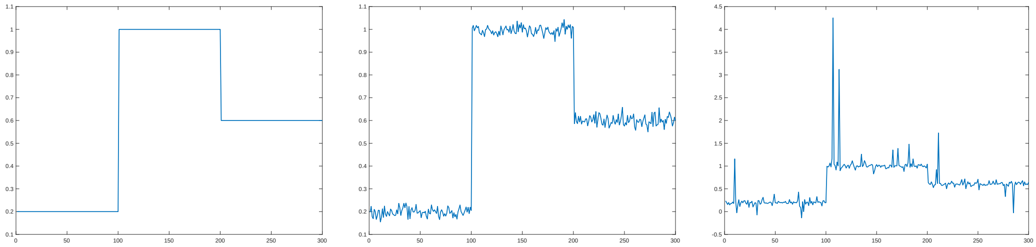

(a) Noise free signal (b) Signal with Gaussian noise (c) Signal with Cauchy noise

Figure 1: A 1D signal degraded by the Gaussian noise with mean 0 and variance 0.02 and the Cauchy noise with $\xi=0.02$.

Cauchy noise. Usually, the probability density function (PDF) of the Cauchy distribution is written as

$$p(x)=\frac{1}{\pi}\frac{\xi}{\xi^2+(x-\delta)^2},$$

where $\xi>0$ and $\delta\in\mathbb{R}$ are called the scale and localization parameters. The scale parameter determines the spread of the distribution around δ, while the localization parameter corresponds to the median of the distribution as discussed in [30]. In this paper, we consider $\delta=0$. From [30, 31], the Cauchy noise can be obtained from the ratio of two independent on Gaussian variables, and thus images corrupted by blur and Cauchy noise can be modeled as

$$f=Hu+\xi\frac{\eta_1}{\eta_2},$$

where $H=[h_1,h_2,\cdots,h_m]$ is a linear blur matrix, $\xi>0$ denotes the noise level, η_1 and η_2 are independent random variables following Gaussian distribution with mean 0 and variation 1. Some recent papers [21, 30, 31, 42] indicate that the Cauchy distribution has the heavy tails characteristic, which leads to noisy images having a greater probability to include outliers than that degraded by the Gaussian distribution. In Fig. 1, we show a simple example where a 1D signal is respectively corrupted by Gaussian noise and Cauchy noise. It is clear that the noisy signal degraded by Cauchy noise is impulsive. Interested readers can refer to more details about the differences between Gaussian, impulsive and Cauchy noise in [21,30,31,42].

A widely studied restoration method is based on variational regularization technique. One of the most popular variational models is ROF model [29] for Gaussian noise removal, where the total variation (TV) regularization term is the ℓ_1 norm of image gradient. Based on TV regularization, Sciacchitano et al. [30] introduced a variational model for recovering images degraded by Cauchy noise and blur as

$$\inf_{u\in BV(\Omega)}\int_\Omega|Du|+\frac{\alpha}{2}\int_\Omega\log(\gamma^2+(Hu-f)^2), \tag{1.1}$$