Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 1, pp. 175-206

DOI: 10.4208/aamm.OA-2023-0004
February 2025

Asymptotic Error Analysis for the Discrete Iterated Galerkin Solution of Urysohn Integral Equations with Green's Kernels

Gobinda Rakshit*

Department of Mathematical Sciences, Rajiv Gandhi Institute of Petroleum Technology (RGIPT), Jais Campus, Amethi, Uttar Pradesh 229304, India

Received 2 January 2023; Accepted (in revised version) 26 June 2023

Abstract. Consider a Urysohn integral equation $x-\mathcal{K}(x)=f$, where f and the integral operator $\mathcal{K}$ with kernel of the type of Green's function are given. In the computation of approximate solutions of the given integral equation by Galerkin method, all the integrals are needed to be evaluated by some numerical integration formula. This gives rise to the discrete version of the Galerkin method. For $r\geq 1$, a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition is chosen to be the approximating space. For the appropriate choice of a numerical integration formula, an asymptotic series expansion of the discrete iterated Galerkin solution is obtained at the above partition points. Richardson extrapolation is used to improve the order of convergence. Using this method we can restore the rate of convergence when the error is measured in the continuous case. Numerical examples are given to illustrate this theory.

AMS subject classifications: 45G10, 65B05, 65J15, 65R20

Key words: Urysohn integral equation, Green's kernel, iterated Galerkin method, Nyström approximation, Richardson extrapolation.

1 Introduction

Let $\mathcal{X}=L^\infty[0,1]$. Consider the problem of solving Urysohn integral equation

$$x(s)-\int_0^1 \kappa(s,t,x(t))dt=f(s), \quad s\in[0,1], \tag{1.1}$$

*Corresponding author.
Email: g.rakshit@rgipt.ac.in (G. Rakshit)

where $f \in \mathcal{X}$ and $\kappa \in C([0,1]\times[0,1]\times\mathbb{R})$ are given. Let the Urysohn integral operator $\mathcal{K}: L^\infty[0,1] \to C[0,1]$ be defined by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))dt, \quad x \in \mathcal{X}, \quad s \in [0,1]. \tag{1.2}$$

Since the kernel κ is continuous, $\mathcal{K}$ is compact operator on $\mathcal{X}$. Denote Eq. (1.1) by

$$x - \mathcal{K}(x) = f. \tag{1.3}$$

We assume that the above equation has a solution, say φ. We also assume that $\mathcal{K}$ is twice Frechét differentiable and 1 is not an eigenvalue of the compact linear operator $\mathcal{K}'(\varphi)$. This gives us that φ is an isolated solution of (1.3). See [12,14]. Note that, if $f \in C^\alpha[0,1]$ for any positive integer α, then $\varphi \in C^\alpha[0,1]$. See [4, Corollary 3.2] and [5, Corollary 4.2]. We are looking for Galerkin approximations of φ.

For $r \geq 1$, consider the approximating space $\mathcal{X}_n$ as a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition, say $\Delta^{(n)}$, of $[0,1]$ with n subintervals each of length $h = \frac{1}{n}$. Let π_n be the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to $\mathcal{X}_n$. Then the Galerkin solution φ_n^G satisfies the following integral equation

$$\varphi_n^G - \pi_n \mathcal{K}(\varphi_n^G) = \pi_n f.$$

Galerkin method for Urysohn integral equation has been studied extensively in research literature. See [12–14] and [4]. The iterated Galerkin solution is defined by

$$\varphi_n^S = \mathcal{K}(\varphi_n^G) + f.$$

In [4], the following orders of convergence are also obtained.

$$\begin{aligned}
&\|\varphi_n^G - \varphi\|_\infty = \mathcal{O}(h), && \|\varphi_n^S - \varphi\|_\infty = \mathcal{O}\left(h^2\right), && \text{if} \quad r = 1, \\
&\|\varphi_n^G - \varphi\|_\infty = \mathcal{O}(h^r), && \|\varphi_n^S - \varphi\|_\infty = \mathcal{O}\left(h^{r+2}\right), && \text{if} \quad r \geq 2.
\end{aligned}$$

It is also shown that the order of convergence of φ_n^S at the points of partition $\Delta^{(n)}$, is h^{2r}.

If an asymptotic expansion for the error exists, one can apply a technique to obtain more accurate approximations. Richardson extrapolation is one of such methods. In [26], an asymptotic expansion for the iterated Galerkin solution of Urysohn integral equation with Green's function type of kernel, is obtained at the above mentioned partition points. Then, by [11] and using Richardson extrapolation, an approximate solution with order of convergence h^{2r+2} can be obtained.

In the computation of above approximations, various integrals are involved. There is an integral in the definition of the Urysohn integral operator $\mathcal{K}$. In the definition of the orthogonal projection π_n, the standard inner product on $L^2[0,1]$ comes into picture. In practice, it is necessary to replace all these integrals by a numerical quadrature formula. This gives rise to the discrete versions of the projection methods. The discrete versions of