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Asymptotic Error Analysis for the Discrete Iterated Galerkin Solution of Urysohn Integral Equations with Green's Kernels

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Abstract. Consider a Urysohn integral equation $x-\mathcal{K}(x)=f$, where f and the integral operator $\mathcal{K}$ with kernel of the type of Green's function are given. In the computation of approximate solutions of the given integral equation by Galerkin method, all the integrals are needed to be evaluated by some numerical integration formula. This gives rise to the discrete version of the Galerkin method. For $r\geq 1$, a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition is chosen to be the approximating space. For the appropriate choice of a numerical integration formula, an asymptotic series expansion of the discrete iterated Galerkin solution is obtained at the above partition points. Richardson extrapolation is used to improve the order of convergence. Using this method we can restore the rate of convergence when the error is measured in the continuous case. Numerical examples are given to illustrate this theory.

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Key words: Urysohn integral equation, Green's kernel, iterated Galerkin method, Nyström approximation, Richardson extrapolation.

1 Introduction

Let $\mathcal{X}=L^{\infty}[0,1]$. Consider the problem of solving Urysohn integral equation

$$x(s)-\int_0^1 \kappa(s,t,x(t))dt=f(s), \quad s\in[0,1], \tag{1.1}$$

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where $f \in \mathcal{X}$ and $\kappa \in C([0,1]\times[0,1]\times\mathbb{R})$ are given. Let the Urysohn integral operator $\mathcal{K}: L^\infty[0,1] \to C[0,1]$ be defined by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))dt, \quad x \in \mathcal{X}, \quad s \in [0,1]. \tag{1.2}$$

Since the kernel κ is continuous, $\mathcal{K}$ is compact operator on $\mathcal{X}$. Denote Eq. (1.1) by

$$x - \mathcal{K}(x) = f. \tag{1.3}$$

We assume that the above equation has a solution, say φ. We also assume that $\mathcal{K}$ is twice Frechét differentiable and 1 is not an eigenvalue of the compact linear operator $\mathcal{K}'(\varphi)$. This gives us that φ is an isolated solution of (1.3). See [12,14]. Note that, if $f \in C^\alpha[0,1]$ for any positive integer α, then $\varphi \in C^\alpha[0,1]$. See [4, Corollary 3.2] and [5, Corollary 4.2]. We are looking for Galerkin approximations of φ.

For $r \geq 1$, consider the approximating space $\mathcal{X}_n$ as a space of piecewise polynomials of degree $\leq r-1$ with respect to a uniform partition, say $\Delta^{(n)}$, of $[0,1]$ with n subintervals each of length $h = \frac{1}{n}$. Let π_n be the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to $\mathcal{X}_n$. Then the Galerkin solution φ_n^G satisfies the following integral equation

$$\varphi_n^G - \pi_n \mathcal{K}(\varphi_n^G) = \pi_n f.$$

Galerkin method for Urysohn integral equation has been studied extensively in research literature. See [12–14] and [4]. The iterated Galerkin solution is defined by

$$\varphi_n^S = \mathcal{K}(\varphi_n^G) + f.$$

In [4], the following orders of convergence are also obtained.

$$\begin{aligned}
&\|\varphi_n^G - \varphi\|_\infty = \mathcal{O}(h), && \|\varphi_n^S - \varphi\|_\infty = \mathcal{O}\left(h^2\right), && \text{if} \quad r = 1,\\
&\|\varphi_n^G - \varphi\|_\infty = \mathcal{O}(h^r), && \|\varphi_n^S - \varphi\|_\infty = \mathcal{O}\left(h^{r+2}\right), && \text{if} \quad r \geq 2.
\end{aligned}$$

It is also shown that the order of convergence of φ_n^S at the points of partition $\Delta^{(n)}$, is h^{2r}.

If an asymptotic expansion for the error exists, one can apply a technique to obtain more accurate approximations. Richardson extrapolation is one of such methods. In [26], an asymptotic expansion for the iterated Galerkin solution of Urysohn integral equation with Green's function type of kernel, is obtained at the above mentioned partition points. Then, by [11] and using Richardson extrapolation, an approximate solution with order of convergence h^{2r+2} can be obtained.

In the computation of above approximations, various integrals are involved. There is an integral in the definition of the Urysohn integral operator $\mathcal{K}$. In the definition of the orthogonal projection π_n, the standard inner product on $L^2[0,1]$ comes into picture. In practice, it is necessary to replace all these integrals by a numerical quadrature formula. This gives rise to the discrete versions of the projection methods. The discrete versions of the Galerkin methods for Urysohn integral with Green's kernel, are investigated in [5,6]. Whereas, in [18,20], a different version of discrete projection method is discussed.

In this article, by discretization, we try to fill the gap between computational result and the theoretical result obtained in [26]. We consider the Urysohn integral equation with Green's kernel, and discrete Galerkin method is applied for approximations. Then, an asymptotic expansion for the discrete iterated Galerkin solution is obtained.

We choose a fine partition (submesh of $\Delta^{(n)}$) of $[0,1]$ with m subintervals each of length $\tilde{h}=\frac{1}{m}$ and define a composite numerical quadrature formula. Replacing the integrals in the definition of $\mathcal{K}$ and π_n, we define the Nyström operator $\mathcal{K}_m$ and the discrete orthogonal projection P_n. Then the discrete Galerkin and the discrete iterated Galerkin equations are given by

$$z_n^G - P_n\mathcal{K}_m(z_n^G) = P_n f \quad \text{and} \quad z_n^S - \mathcal{K}_m(P_n z_n^S) = f,$$

respectively. If $\varphi \in C^{r+2}[0,1]$, then from [5] and [18], we have

$$\left\| z_n^G - \varphi \right\|_\infty = \mathcal{O}\left(\max\left\{h^r, \tilde{h}^2\right\}\right), \tag{1.4a}$$

$$\left\| z_n^S - \varphi \right\|_\infty = \begin{cases} \mathcal{O}\left(\max\left\{h^2, \tilde{h}^2\right\}\right), & r=1, \\ \mathcal{O}\left(\max\left\{h^{r+2}, \tilde{h}^2\right\}\right), & r \geq 2. \end{cases} \tag{1.4b}$$

In [18], discrete modified projection methods are applied to (1.1) and an improved order of convergence for the discrete iterated modified projection solution (z_n^M) is achieved, which is as follows.

$$\left\| z_n^M - \varphi \right\|_\infty = \begin{cases} \mathcal{O}\left(\max\left\{h^4, \tilde{h}^2\right\}\right), & r=1, \\ \mathcal{O}\left(\max\left\{h^{r+4}, \tilde{h}^2\right\}\right), & r \geq 2. \end{cases}$$

In this article, first we find an asymptotic error expansion due to the discrete orthogonal projection. Then using this, the following asymptotic expansion is obtained:

$$z_n^S(t_i) = \varphi(t_i) + \gamma(t_i)h^{2r} + \mathcal{O}\left(\max\left\{h^{2r+2}, \tilde{h}^2\right\}\right), \tag{1.5}$$

where the function γ is independent of h. If we choose m such that $\tilde{h} \leq h^{2r+2}$, then using the Richardson extrapolation, an approximation of φ could be obtained of the order h^{2r+2}, which is better than the order of discrete iterated modified projection solution.

This article is organized as follows. Definitions, notations and some preliminary results are given in Section 2. In Section 3, a quadrature rule is deduced, and using it the discrete orthogonal projection and the Nyström approximations of the integral operators are defined. Section 4 contains the asymptotic error analysis for the approximations. A Numerical example is given in Section 5.

2 Preliminaries

For an integer $\alpha \geq 0$, let $C^\alpha[0,1]$ denotes the space of all real valued α-times continuously differentiable functions on $[0,1]$ with the norm

$$\|x\|_{\alpha,\infty} = \max_{0\leq j\leq \alpha} \left\|x^{(j)}\right\|_\infty,$$

where $x^{(j)}$ is the j^{th} derivative of the function x, and

$$\left\|x^{(j)}\right\|_\infty = \sup_{0\leq t\leq 1} |x^{(j)}(t)|.$$

Define

$$\|\kappa\|_{\alpha,\infty} = \max_{0\leq i+j+k\leq \alpha} \left\|D^{(i,j,k)}\kappa(s,t,u)\right\|_\infty,$$

where

$$D^{(i,j,k)}\kappa(s,t,u) = \frac{\partial^{i+j+k}\kappa}{\partial s^i \partial t^j \partial u^k}(s,t,u).$$

2.1 Green's function type kernel

Let $r \geq 1$ be an integer and assume that the kernel κ has the following properties.

1. For $i = 1,2,3,4$, the functions $\kappa, \frac{\partial^i \kappa}{\partial u^i} \in C(\Omega)$, where $C(\Omega)$ denotes the space of all real valued continuous function on $\Omega = [0,1]\times[0,1]\times\mathbb{R}$.
2. Let $\Omega_1 = \{(s,t,u) : 0 \leq t \leq s \leq 1, u \in \mathbb{R}\}$ and $\Omega_2 = \{(s,t,u) : 0 \leq s \leq t \leq 1, u \in \mathbb{R}\}$. There are two functions $\kappa_j \in C^r(\Omega_j), j = 1,2$, such that
$$\kappa(s,t,u) = \begin{cases} \kappa_1(s,t,u), & (s,t,u) \in \Omega_1, \\ \kappa_2(s,t,u), & (s,t,u) \in \Omega_2. \end{cases}$$
3. Denote
$$\ell(s,t,u) = \frac{\partial \kappa}{\partial u}(s,t,u) \quad \text{and} \quad \lambda(s,t,u) = \frac{\partial^2 \kappa}{\partial u^2}(s,t,u),$$
for all $(s,t,u) \in \Omega$. The partial derivatives of $\ell(s,t,u)$ and $\lambda(s,t,u)$ with respect to s and t have jump discontinuities on $s = t$.
4. There are functions $\ell_j, \lambda_j \in C^r(\Omega_j)$, $j = 1,2$, with
$$\ell(s,t,u) = \begin{cases} \ell_1(s,t,u), & (s,t,u) \in \Omega_1, \\ \ell_2(s,t,u), & (s,t,u) \in \Omega_2, \end{cases} \qquad \lambda(s,t,u) = \begin{cases} \lambda_1(s,t,u), & (s,t,u) \in \Omega_1, \\ \lambda_2(s,t,u), & (s,t,u) \in \Omega_2. \end{cases}$$

The kernels in the following two examples satisfy the above four properties.

(i) Consider the integral equation of the second kind

$$x(s)-\int_0^1 k_c(s,t)\left[c^2x(t)-g(t,x(t))\right]dt=f(s),\quad 0\leq s\leq 1,$$

where

$$k_c(s,t)=\frac{1}{c\sinh(\gamma)}\begin{cases}\sinh(cs)\sinh(c(1-t)), & 0\leq s<t,\\ \sinh(c(1-s))\sinh(ct), & t\leq s\leq 1,\end{cases}$$

$f(s)=\frac{1}{\sinh(c)}[a\sinh(c(1-s))+b\sinh(cs)]$ and c is a parameter chosen such a way that the Picard iteration converge. This equation can be derived (cf. [24, Example 2]) from the two-point boundary value problem

$$\begin{aligned}&x''(s)-g(s,x(s))=0,\quad 0<s<1,\\&x(0)=a,\quad x(1)=b.\end{aligned}$$

(ii) Consider the non-linear integral equation

$$x(s)=\int_0^1 k(s,t)\left[g(t,x(t))+f(t)\right]dt,$$

where

$$k(s,t)=\begin{cases}t(s-1), & 0\leq t\leq s,\\ s(t-1), & s\leq t\leq 1,\end{cases}$$

$g(t,u)=\frac{1}{1+t+u}$ and the function f is so chosen such that $\varphi(s)=\frac{s(1-s)}{1+s}$ is a solution of the above problem. This integral equation can be reformulated (cf. [4]) from the boundary value problem

$$\begin{aligned}&x''(s)=g(s,x(s))+f(s),\quad 0<s<1,\\&x'(0)=0,\quad x(1)=0.\end{aligned}$$

Under the above assumptions (since $\frac{\partial^i\kappa}{\partial u^i}\in C(\Omega)$ for $i=1,2,3,4$), the operator $\mathcal{K}$ is four times Fréchet differentiable, and its Fréchet derivatives at $x\in\mathcal{X}$ are given by

$$\mathcal{K}^{(i)}(x)(v_1,\cdots,v_i)(s)=\int_0^1\frac{\partial^i\kappa}{\partial u^i}(s,t,x(t))v_1(t)\cdots v_i(t)dt,\quad i=1,2,3,4,$$

where

$$\frac{\partial^i\kappa}{\partial u^i}(s,t,x(t))=\frac{\partial^i\kappa}{\partial u^i}(s,t,u)\Big|_{u=x(t)},\quad i=1,2,3,4,$$

and $v_1,v_2,v_3,v_4 \in \mathcal{X}$. Note that $\mathcal{K}'(x):\mathcal{X}\to\mathcal{X}$ is linear and $\mathcal{K}^{(i)}(x):\mathcal{X}^i\to\mathcal{X}$ are multi-linear operators, where $\mathcal{X}^i$ is the cartesian product of i copies of $\mathcal{X}$. See [27]. The norms of these operators are defined by

$$\left\|\mathcal{K}^{(i)}(x)\right\| = \sup\left\{\left\|\mathcal{K}^{(i)}(x)(v_1,\cdots,v_i)\right\|_\infty : \|v_j\|_\infty \le 1,\ j=1,\cdots,i\right\}$$

for $i=1,2,3,4$. It follows that

$$\left\|\mathcal{K}^{(i)}(x)\right\| \le \sup_{0\le s,t\le 1}\left|\frac{\partial^i \kappa}{\partial u^i}(s,t,x(t))\right|, \quad i=1,2,3,4.$$

Let

$$\mathcal{K}'(\varphi)v(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,\varphi(t))v(t)dt = \int_0^1 \ell(s,t,\varphi(t))v(t)dt, \quad v\in\mathcal{X}. \tag{2.1}$$

Then $\mathcal{K}'(\varphi)$ is a compact linear operator.

3 Discretization of integrals by numerical quadrature rule

In this section, first we consider a numerical integration formula. We replace the integral in the standard inner product of $L^2[0,1]$ (i.e., $\langle x,y\rangle = \int_0^1 x(t)y(t)dt$) by the quadrature rule and define a discrete inner product. Subsequently, the corresponding discrete orthogonal projection is defined. After that, an asymptotic error expansion for the discrete orthogonal projection is obtained. Next we define the Nyström approximations of the integral operator $\mathcal{K}$ and its Fréchet derivatives.

Consider a basic numerical integration formula

$$\int_0^1 x(t)dt \approx \sum_{q=1}^{\rho} w_q x(\mu_q), \tag{3.1}$$

where w_q and μ_q denote the weights and quadrature nodes, respectively. Assume that the quadrature rule (3.1) is exact for polynomials of degree $\le 3r$, and that it is exact at least for linear polynomials when $r=0$. So we have $\sum_{q=1}^{\rho} w_q = 1$.

Let $n\in\mathbb{N}$ and consider the following uniform partition of $[0,1]$:

$$\Delta^{(n)}: \quad 0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1. \tag{3.2}$$

Define $t_j = \frac{j}{n}$, $\Delta_j = [t_{j-1},t_j]$ and $h = t_j - t_{j-1} = \frac{1}{n}$, $j=1,\cdots,n$. Define the subspace $C^{\alpha}_{\Delta^{(n)}}[0,1] = \left\{x\in\mathcal{X} : x\in C^{\alpha}[t_{j-1},t_j],\ j=1,2,3,\cdots,n\right\}$. For $r\ge 1$, the approximating space

$$\mathcal{X}_n = \left\{x\in\mathcal{X} : x|_{\Delta_j} \text{is a polynomial of degree } \le r-1\right\}.$$

Let p be a positive integer and $m=pn$. Consider the following uniform partition of $[0,1]$:

$$\Delta^{(m)}: \quad 0<\frac{1}{m}<\cdots<\frac{m-1}{m}<1. \tag{3.3}$$

Let $\tilde{h}=\frac{1}{m}$ and $s_i=\frac{i}{m}$, $i=0,\cdots,m$.

Please note that: As our goal to find Eq. (1.5), where the higher order term is maximum of h^{2r+2} and $\tilde{h}^2$, we choose the partition $\Delta^{(m)}$ such that $\tilde{h}^2\le h^r$.

A composite integration rule with respect to the partition (3.3) is then defined as

$$\int_0^1 x(t)dt=\sum_{i=1}^{m}\int_{s_{i-1}}^{s_i} x(t)dt\approx\tilde{h}\sum_{i=1}^{m}\sum_{q=1}^{\rho} w_q x(s_{i-1}+\mu_q\tilde{h}).$$

Thus,

$$\int_{t_{j-1}}^{t_j} x(t)dt=\int_{(j-1)h}^{jh} x(t)dt=\sum_{\nu=1}^{p}\int_{(j-1)h+(\nu-1)\tilde{h}}^{(j-1)h+\nu\tilde{h}} x(t)dt.$$

Since $h=p\tilde{h}$,

$$\int_{t_{j-1}}^{t_j} x(t)dt=\sum_{\nu=1}^{p}\int_{\frac{(j-1)p+\nu-1}{p}h}^{\frac{(j-1)p+\nu}{p}h} x(t)dt.$$

Substituting

$$t=\frac{(j-1)p+\nu-1}{p}h+\tilde{h}\sigma=\frac{(j-1)p+\nu-1+\sigma}{p}h$$

in the above equation, we obtain

$$\begin{aligned}\int_{t_{j-1}}^{t_j} x(t)dt&=\frac{h}{p}\sum_{\nu=1}^{p}\int_0^1 x\left(\frac{(j-1)p+\nu-1+\sigma}{p}h\right)d\sigma\\&=\frac{h}{p}\sum_{\nu=1}^{p}\int_0^1 x\left(t_{j-1}+\frac{\nu-1+\sigma}{p}h\right)d\sigma.\end{aligned}$$

Note that $\frac{\nu-1+\sigma}{p}\in[0,1]$. Now using the numerical quadrature formula (3.1), we obtain

$$\int_{t_{j-1}}^{t_j} x(t)dt\approx\frac{h}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q x\left(t_{j-1}+\frac{\nu-1+\mu_q}{p}h\right).$$

Let

$$\mu_{q\nu}=\frac{\nu-1+\mu_q}{p}, \quad q=1,2,\cdots,\rho; \quad \nu=1,2,\cdots,p.$$

Then,

$$\int_{t_{j-1}}^{t_j} x(t)dt \approx \frac{h}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q x\big(t_{j-1}+\mu_{q\nu}h\big). \tag{3.4}$$

We prove the following lemma which will be used to find an asymptotic error expansion for the discrete orthogonal projection.

Lemma 3.1. *Let L_η be the Legendre polynomial of degree $\eta \in \{0,1,\cdots,r-1\}$ defined on $[0,1]$. Then for any $k=1,2,\cdots,2r-1$,*

$$\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta(\mu_{q\nu})L_\eta(\tau)\frac{\big(\mu_{q\nu}-\tau\big)^k}{k!} = \int_0^1 \Lambda_r(\tau,s)\frac{(s-\tau)^k}{k!}ds,$$

where

$$\sum_{\eta=0}^{r-1} L_\eta(\tau)L_\eta(s) = \Lambda_r(\tau,s) \quad \textit{for} \;\; \tau,s\in[0,1].$$

Proof. Since L_η is a polynomial of degree $0\le\eta\le r-1$,

$$\begin{aligned}&\frac{1}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta\left(\frac{\nu-1+\mu_q}{p}\right) = \frac{1}{p}\sum_{\nu=1}^{p}\int_0^1 L_\eta\left(\frac{\nu-1+t}{p}\right)dt \\ =&\sum_{\nu=1}^{p}\int_{\frac{\nu-1}{p}}^{\frac{\nu}{p}} L_\eta(s)ds = \int_0^1 L_\eta(s)ds.\end{aligned}$$

Since the basic quadrature formula (3.1) is exact for polynomials of degree $\le 3r$,

$$\frac{1}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta\left(\frac{\nu-1+\mu_q}{p}\right)\left(\frac{\nu-1+\mu_q}{p}-\tau\right)^k = \int_0^1 L_\eta(s)(s-\tau)^k ds.$$

It follows that

$$\frac{1}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta(\mu_{q\nu})\frac{\big(\mu_{q\nu}-\tau\big)^k}{k!} = \int_0^1 L_\eta(s)\frac{(s-\tau)^k}{k!}ds,$$

where $\mu_{q\nu}=\frac{\nu-1+\mu_q}{p}$. This gives

$$\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta(\tau)L_\eta(\mu_{q\nu})\frac{\big(\mu_{q\nu}-\tau\big)^k}{k!} = \sum_{\eta=0}^{r-1} L_\eta(\tau)\int_0^1 L_\eta(s)\frac{(s-\tau)^k}{k!}ds.$$

Let

$$\sum_{\eta=0}^{r-1} L_\eta(\tau)L_\eta(s) = \Lambda_r(\tau,s), \quad \tau,s\in[0,1].$$

Then,

$$\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q L_\eta(\tau) L_\eta(\mu_{q\nu})\frac{(\mu_{q\nu}-\tau)^k}{k!} = \int_0^1 \Lambda_r(\tau,s)\frac{(s-\tau)^k}{k!}ds.$$

Hence the required result follows. □

3.1 Discrete orthogonal projection

Let $j\in\{1,2,\cdots,n\}$ and $x,y\in C(\Delta_j)$. Define a discrete inner product on Δ_j by

$$\langle x,y\rangle_{\Delta_j,m} = \tilde{h}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho} w_q x(t_{j-1}+\mu_{q\nu}h) y(t_{j-1}+\mu_{q\nu}h). \tag{3.5}$$

Note that, this is an indefinite inner product. For more details on indefinite inner product spaces, see [8]. However, the properties which we need to define a discrete orthogonal projection, hold true for (3.5). For $\eta=0,1,\cdots,r-1$, let L_η denote the Legendre polynomial of degree η on $[0,1]$. For $j=2,\cdots,n$, and for $\eta=0,1,\cdots,r-1$, define

$$\varphi_{j,\eta}(t) = \begin{cases} \sqrt{\dfrac{1}{h}} L_\eta\left(\dfrac{t-t_{j-1}}{h}\right), & t\in(t_{j-1},t_j], \\ 0, & \text{otherwise}, \end{cases}$$

and,

$$\varphi_{1,\eta}(t) = \sqrt{\frac{1}{h}} L_\eta\left(\frac{t-t_0}{h}\right)$$

if $t\in[t_0,t_1]$ and 0 otherwise. Note that

$$\varphi_{j,\eta}(t_{j-1}+\mu_{q\nu}h) = h^{-\frac{1}{2}} L_\eta(\mu_{q\nu}) \quad \text{for all} \quad j=1,2,\cdots,n. \tag{3.6}$$

Note that $\{\varphi_{j,\eta}: j=1,\cdots,n, \eta=0,1,\cdots,r-1\}$ is a set of orthonormal basis for $\mathcal{X}_n$, where $\varphi_{j,\eta}$ is the Legendre polynomial of degree η defined on $[t_{j-1},t_j]$. Since the basic numerical integration (3.1) has degree of precision $3r$, the set $\{\varphi_{j,\eta}\}$ is also orthonormal with respect to the discrete inner product (3.5). Define the discrete orthogonal projection $P_{n,j}: C[t_{j-1},t_j] \to \mathbb{P}_{r,\Delta_j}$ as follows:

$$P_{n,j}x = \sum_{\eta=0}^{r-1} \langle x, \varphi_{j,\eta}\rangle_{\Delta_j,m} \varphi_{j,\eta}. \tag{3.7}$$

See [5,6] for more details. A discrete orthogonal projection $P_n: C[0,1]\to\mathcal{X}_n$ is defined by

$$P_n x = \sum_{j=1}^{n} P_{n,j} x. \tag{3.8}$$

It follows that $P_n x(t) = P_{n,j} x(t)$, for all $t \in [t_{j-1}, t_j]$. It can be also shown that $\|P_n\| < \infty$ and, if $x \in C^r[t_{j-1}, t_j]$, then

$$\left\|x - P_{n,j} x\right\|_{\Delta_j,\infty} \leq C_1 \left\|x^{(r)}\right\|_{\Delta_j,\infty} h^r, \tag{3.9}$$

if $x \in C^r[0,1]$, then

$$\|x - P_n x\|_{\Delta_j,\infty} \leq C_1 \left\|x^{(r)}\right\|_{\infty} h^r, \tag{3.10}$$

where

$$\|x\|_{\Delta_j,\infty} = \sup_{t \in [t_{j-1}, t_j]} |x(t)|$$

and, C_1 is a constant independent of h. For details see [18].

In (3.10) we have an error bound for the discrete orthogonal projection. But, by the following lemma we obtain an asymptotic error expansion for the discrete orthogonal projection, which is a more stronger result than (3.10).

Lemma 3.2. *Let P_n be the discrete orthogonal projection defined by* (3.7)-(3.8). *Let $x \in C^{2r+2}_{\Delta^{(n)}}[0,1]$ and $t = t_{j-1} + \tau h$ with $\tau \in [0,1]$. Then*

$$P_n x(t) - x(t) = \sum_{k=1}^{2r+1} J_k(\tau) x^{(k)}(t_{j-1} + \tau h) h^k + \mathcal{O}\left(h^{2r+2}\right),$$

where

$$J_k(\tau) = \int_0^1 \Lambda_r(\tau, s) \frac{(s-\tau)^k}{k!} ds, \quad k = 1, 2, \cdots, 2r+1.$$

Proof. Define a function $v_j : [t_{j-1}, t_j] \to \mathbb{R}$ by

$$v_j(t) = 1, \quad t \in [t_{j-1}, t_j].$$

For $\tau \in [0,1]$, let $t = t_{j-1} + h\tau \in [t_{j-1}, t_j]$. From (3.7) it is easy to see that

$$P_{n,j} v_j = v_j.$$

It follows that

$$\sum_{\eta=0}^{r-1} \left\langle v_j, \varphi_{j,\eta} \right\rangle_{\Delta_j,m} \varphi_{j,\eta}(t) = 1.$$

Since $\varphi_{j,\eta}$ is a polynomial of degree $0 \leq \eta \leq r-1$ on $[t_{j-1}, t_j]$,

$$\left\langle v_j, \varphi_{j,\eta} \right\rangle_{\Delta_j} = \int_{t_{j-1}}^{t_j} \varphi_{j,\eta}(s) ds = \frac{h}{p} \sum_{\nu=1}^{p} \sum_{q=1}^{\rho} w_q \varphi_{j,\eta}\left(t_{j-1} + \mu_{q\nu} h\right).$$

Thus for any function $x:[t_{j-1},t_j]\to\mathbb{R}$, we have

$$x(t)=x(t)\frac{h}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_q\varphi_{j,\eta}\left(t_{j-1}+\mu_{q\nu}h\right)\varphi_{j,\eta}(t).$$

It follows that

$$P_{n,j}x(t)-x(t)=\frac{h}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_q\varphi_{j,\eta}\left(t_{j-1}+\mu_{q\nu}h\right)\left[x(t_{j-1}+\mu_{q\nu}h)-x(t)\right]\varphi_{j,\eta}(t),$$

where $t=t_{j-1}+h\tau\in[t_{j-1},t_j]$ and $\tau\in[0,1]$. From (3.6), we have

$$P_{n,j}x(t)-x(t)=\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_qL_\eta(\mu_{q\nu})\left[x(t_{j-1}+\mu_{q\nu}h)-x(t)\right]L_\eta(\tau). \tag{3.11}$$

Since $x\in C^{2r+2}[t_{j-1},t_j]$, using Taylor series expansion we obtain

$$x(t_{j-1}+\mu_{q\nu}h)-x(t_{j-1}+h\tau)=\sum_{k=1}^{2r+1}x^{(k)}(t_{j-1}+\tau h)\frac{\left(\mu_{q\nu}-\tau\right)^k}{k!}h^k+\mathcal{O}\left(h^{2r+2}\right).$$

Thus

$$\begin{aligned}&P_nx(t)-x(t)=P_{n,j}x(t)-x(t)\\=&\sum_{k=1}^{2r+1}x^{(k)}(t_{j-1}+\tau h)h^k\left\{\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_qL_\eta(\mu_{q\nu})L_\eta(\tau)\frac{\left(\mu_{q\nu}-\tau\right)^k}{k!}\right\}+\mathcal{O}\left(h^{2r+2}\right).\end{aligned}$$

Let

$$J_k(\tau)=\frac{1}{p}\sum_{\eta=0}^{r-1}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_qL_\eta(\mu_{q\nu})L_\eta(\tau)\frac{\left(\mu_{q\nu}-\tau\right)^k}{k!},\quad \tau\in[0,1].$$

By Lemma 3.1, we can write

$$J_k(\tau)=\int_0^1\Lambda_r(\tau,s)\frac{(s-\tau)^k}{k!}ds.$$

Hence

$$P_nx(t)-x(t)=\sum_{k=1}^{2r+1}J_k(\tau)x^{(k)}(t_{j-1}+\tau h)h^k+\mathcal{O}\left(h^{2r+2}\right).$$

The result follows. □

Let

$$\mathcal{L}=\left(I-\mathcal{K}'(\varphi)\right)^{-1}\mathcal{K}'(\varphi).$$

Then $\mathcal{L}$ is a compact linear Fredholm integral operator of the second kind (see [21]). Let $\tilde{\ell}$ be the kernel of $\mathcal{L}$. Since ℓ (in page 4 and (2.1)) is Green's function type of kernel, $\tilde{\ell}$ is also a Green's function type of kernel (see [4,28, Lemma 5.1]). It follows that

$$\mathcal{L}P_n x(s)-\mathcal{L}x(s)=\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\tilde{\ell}(s,t)\left(P_n x(t)-x(t)\right)dt,\quad \forall x\in\mathcal{X}.$$

Then using Lemma 3.2, and following the proofs of [23, Theorem 5.1] and [25, Theorem 3.2], it can be shown that

$$\mathcal{L}(I-P_n)\varphi(t_i)=\mathcal{E}_{2r}(\varphi)(t_i)h^{2r}+\mathcal{O}\left(h^{2r+2}\right),\quad i=0,1,\cdots,n, \tag{3.12}$$

where

$$\begin{aligned}\mathcal{E}_{2r}(\varphi)(t_i)=&\bar{b}_{2r,2r}\int_0^1\tilde{\ell}(t_i,t)(t)\varphi^{(2r)}(t)dt+\sum_{p=1}^{2r-1}\bar{b}_{2r,p}\Bigg\{\left[\left(\frac{\partial}{\partial t}\right)^{2r-p-1}\left(\tilde{\ell}(t_i,t)\varphi^{(p)}(t)\right)\right]_{t=0}^{t=1}\\&-\left[\left(\frac{\partial}{\partial t}\right)^{2r-p-1}\left(\tilde{\ell}(t_i,t)\varphi^{(p)}(t)\right)\right]_{t=t_i-}^{t=t_i+}\Bigg\}\end{aligned}$$

with

$$\bar{b}_{2r,p}=\int_0^1\int_0^1\Lambda_r(\tau,s)\frac{(\tau-s)^p}{p!}\frac{B_{2r-p}(s)}{(2r-p)!}d\tau ds$$

and B_k is the Bernoulli polynomial of degree $k\geq 0$.

3.2 Approximation of the integral operator

Let $x\in\mathcal{X}$. Recall that

$$\mathcal{K}(x)(s)=\int_0^1\kappa(s,t,x(t))dt,\quad s\in[0,1].$$

Replacing the above integral by the numerical quadrature rule (3.4), we define the Nyström approximation of $\mathcal{K}$ by

$$\mathcal{K}_m(x)(s)=\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\,\kappa\left(s,t_{j-1}+\mu_{q\nu}h,x\left(t_{j-1}+\mu_{q\nu}h\right)\right),\quad s\in[0,1].$$

Let $\left\{\mu_{q\nu}^j=t_{j-1}+\mu_{q\nu}h: j=1,2,\cdots,n;\ q=1,2,\cdots,\rho;\ \nu=1,2,\cdots,p\right\}$ be the set of all quadrature nodes in [0,1]. Then

$$\mathcal{K}_m(x)(s)=\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\,\kappa\left(s,\mu_{q\nu}^j,x\left(\mu_{q\nu}^j\right)\right),\quad s\in[0,1]. \tag{3.13}$$

The Nyström method for solving (1.1) is to solve

$$x-\mathcal{K}_m(x)=f.$$

For sufficiently large m, the above equation has a unique solution φ_m in a neighborhood $B(\varphi,\epsilon)$ of φ, and

$$\|\varphi-\varphi_m\|_\infty\leq C_2\|\mathcal{K}(\varphi)-\mathcal{K}_m(\varphi)\|_\infty=\mathcal{O}\left(\tilde{h}^2\right), \tag{3.14}$$

where C_2 is a constant independent of m. See [1, Theorem 4]. We write

$$(I-P_n)\varphi_m=(I-P_n)(\varphi_m-\varphi)+(I-P_n)\varphi.$$

Then from (3.10), (3.14), we have

$$(I-P_n)\varphi_m=\mathcal{O}\left(\max\left\{h^r,\tilde{h}^2\right\}\right). \tag{3.15}$$

Let $v_1,v_2\in\mathcal{X}$ and $x\in B(\varphi,\epsilon)$. Then the Fréchet derivatives of $\mathcal{K}_m$ at x are given by

$$\begin{aligned}\mathcal{K}_m'(x)v_1(s)&=\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\,D^{(0,0,1)}\kappa\left(s,\mu_{q\nu}^j,x\left(\mu_{q\nu}^j\right)\right)v_1\left(\mu_{q\nu}^j\right),\quad s\in[0,1],\\ \mathcal{K}_m''(x)(v_1,v_2)(s)&=\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\,\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x\left(\mu_{q\nu}^j\right)\right)v_1\left(\mu_{q\nu}^j\right)v_2\left(\mu_{q\nu}^j\right).\end{aligned}$$

It follows that

$$\left\|\mathcal{K}_m''(x)(v_1,v_2)\right\|_\infty\leq\left(\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\epsilon}}\left|\frac{\partial^2\kappa}{\partial u^2}(s,t,u)\right|\right)\|v_1\|_\infty\|v_2\|_\infty.$$

This implies

$$\left\|\mathcal{K}_m''(x)\right\|\leq\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\epsilon}}\left|\frac{\partial^2\kappa}{\partial u^2}(s,t,u)\right|<\infty.$$

Similarly, it can be shown that

$$\left\|\mathcal{K}_m^{(3)}(x)\right\|\leq\left(\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\epsilon}}\left|\frac{\partial^3\kappa}{\partial u^3}(s,t,u)\right|\right)=C_3<\infty.$$

Lemma 3.3. *Let $x_1,x_2\in B(\varphi,\epsilon)$. If $D^{(0,0,3)}\kappa\in C(\Omega)$ then*

$$\left\|\mathcal{K}_m''(x_1)-\mathcal{K}_m''(x_2)\right\|\leq C_3\|x_1-x_2\|_\infty,$$

where C_3 is constant independent of n.

Proof. For $v_1,v_2\in\mathcal{X}$, we have

$$\begin{aligned}&\left[\mathcal{K}_m''(x_1)-\mathcal{K}_m''(x_2)\right](v_1,v_2)(s)\\ =&\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\left[\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_1\left(\mu_{q\nu}^j\right)\right)-\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_2\left(\mu_{q\nu}^j\right)\right)\right]v_1\left(\mu_{q\nu}^j\right)v_2\left(\mu_{q\nu}^j\right)\end{aligned}$$

for all $s\in[0,1]$. Since $D^{(0,0,3)}\kappa\in C(\Omega)$, applying mean value theorem on $\frac{\partial^2\kappa}{\partial u^2}$ with respect to its third variable u, we obtain

$$\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_1\left(\mu_{q\nu}^j\right)\right)-\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_2\left(\mu_{q\nu}^j\right)\right)=\left[x_1\left(\mu_{q\nu}^j\right)-x_2\left(\mu_{q\nu}^j\right)\right]\frac{\partial^3\kappa}{\partial u^3}\left(s,\mu_{q\nu}^j,\zeta_{q\nu}^j\right),$$

where $\zeta_{q\nu}^j$ lies in the line segment joining the points $x_1\left(\mu_{q\nu}^j\right)$ and $x_2\left(\mu_{q\nu}^j\right)$. Then

$$\left|\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_1\left(\mu_{q\nu}^j\right)\right)-\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^j,x_2\left(\mu_{q\nu}^j\right)\right)\right|\le C_3\|x_1-x_2\|_\infty.$$

Hence

$$\left\|\left[\mathcal{K}_m''(x_1)-\mathcal{K}_m''(x_2)\right](v_1,v_2)\right\|_\infty\le C_3\|x_1-x_2\|_\infty\|v_1\|_\infty\|v_2\|_\infty,$$

which follows the result. □

We will now quote some error estimates for the Nyström approximations.

For $\alpha\ge 0$, if $v_1,v_2\in C^{\alpha}_{\Delta^{(m)}}[0,1]$, then from [29] or [6, Corollary 1], we obtain the following errors for numerical integration.

$$\left\|\left[\mathcal{K}_m'(\varphi)-\mathcal{K}'(\varphi)\right]v_1\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\right), \tag{3.16}$$

and

$$\left\|\left[\mathcal{K}_m''(\varphi)-\mathcal{K}''(\varphi)\right](v_1,v_2)\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\right).$$

Also from [17, Proposition 3.3], we have

$$\left\|\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\right\|\le C_4\|\varphi_m-\varphi\|_\infty=\mathcal{O}\left(\tilde{h}^2\right). \tag{3.17}$$

Therefore combining (3.16) and the above equation, we obtain

$$\left\|\left[\mathcal{K}_m'(\varphi_m)-\mathcal{K}'(\varphi)\right]v\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\right)\quad\text{for all }\ v\in C^{\nu}_{\Delta_m}[0,1]. \tag{3.18}$$

Similarly,

$$\left\|\left[\mathcal{K}_m''(\varphi_m)-\mathcal{K}''(\varphi)\right](v_1,v_2)\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\right),\quad\forall v_1,v_2\in C^{\nu}_{\Delta_m}[0,1], \tag{3.19}$$

for all $v_1,v_2,v_3\in C^{\nu}_{\Delta_m}[0,1]$ implies

$$\left\|\left[\mathcal{K}_m^{(3)}(\varphi_m)-\mathcal{K}^{(3)}(\varphi)\right](v_1,v_2,v_3)\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\right). \tag{3.20}$$

4 Asymptotic error analysis

Replacing $\mathcal{K}$ by $\mathcal{K}_m$ and π_n by P_n in the Galerkin equation $x-\pi_n\mathcal{K}(x)=\pi_n f$, the discrete Galerkin equation is defined by

$$z_n^G - P_n\mathcal{K}_m(z_n^G) = P_n f,$$

where z_n^G is the discrete Galerkin solution. Then the discrete iterated Galerkin solution is defined by

$$z_n^S = \mathcal{K}_m(z_n^G) + f.$$

Note that $P_n z_n^S = z_n^G$. From the equations

$$\varphi_m - \mathcal{K}_m(\varphi_m) = f \quad \text{and} \quad z_n^S - \mathcal{K}_m(P_n z_n^S) = f,$$

we obtain the following error term.

$$\begin{aligned} z_n^S - \varphi_m = & \left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right] \\ & -\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right] \\ & -\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)-\mathcal{L}_m(I-P_n)\varphi_m, \end{aligned} \tag{4.1}$$

where

$$\mathcal{L}_m = \left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m'(\varphi_m).$$

Using the Resolvent Identity, we get

$$\begin{aligned} & \left(I-\mathcal{K}_m'(\varphi_m)\right)^{-1} - \left(I-\mathcal{K}'(\varphi)\right)^{-1} \\ = & \left(I-\mathcal{K}'(\varphi)\right)^{-1}\left[\mathcal{K}_m'(\varphi_m)-\mathcal{K}'(\varphi)\right]\left(I-\mathcal{K}_m'(\varphi_m)\right)^{-1}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(I-\mathcal{K}_m'(\varphi_m)\right)^{-1} = & \left(I-\mathcal{K}'(\varphi)\right)^{-1} + \left(I-\mathcal{K}'(\varphi)\right)^{-1}\left[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\right]\left(I-\mathcal{K}_m'(\varphi_m)\right)^{-1} \\ & + \left(I-\mathcal{K}'(\varphi)\right)^{-1}\left[\mathcal{K}_m'(\varphi)-\mathcal{K}'(\varphi)\right]\left(I-\mathcal{K}_m'(\varphi_m)\right)^{-1}. \end{aligned} \tag{4.2}$$

Now, we will analyze each of the terms appearing in the RHS of Eq. (4.1). Error estimates for each of the referred terms will be obtained by the following propositions.

Proposition 4.1. *Let* $\{t_i : i=0,1,\cdots,n\}$ *be the set of partition points of* $[0,1]$ *defined by* (3.2), *then*

$$\mathcal{L}_m(I-P_n)\varphi_m(t_i) = \mathcal{E}_{2r}(\varphi)(t_i)h^{2r} + \mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right),$$

where $\mathcal{E}_{2r}$ *is defined by* (3.12).

Proof. It can be easily verified that (using (4.2))

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\varphi_m=\big[I-\mathcal{K}_m'(\varphi_m)\big]^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\\ =&\big(I-\mathcal{K}'(\varphi)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\\ &+\big(I-\mathcal{K}'(\varphi)\big)^{-1}\big[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\big]\big(I-\mathcal{K}_m'(\varphi_m)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\\ &+\big(I-\mathcal{K}'(\varphi)\big)^{-1}\big[\mathcal{K}_m'(\varphi)-\mathcal{K}'(\varphi)\big]\big(I-\mathcal{K}_m'(\varphi_m)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m. \end{aligned}\tag{4.3}$$

Consider the first term of the above equation, we have

$$\begin{aligned}&\big(I-\mathcal{K}'(\varphi)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\\ =&\big(I-\mathcal{K}'(\varphi)\big)^{-1}\mathcal{K}'(\varphi)(I-P_n)\varphi+\big(I-\mathcal{K}'(\varphi)\big)^{-1}\mathcal{K}'(\varphi)(I-P_n)(\varphi_m-\varphi)\\ &+\big(I-\mathcal{K}'(\varphi)\big)^{-1}\big[\mathcal{K}_m'(\varphi_m)-\mathcal{K}'(\varphi)\big](I-P_n)\varphi_m.\end{aligned}$$

Using (3.12), (3.14) and (3.18), we obtain

$$\big(I-\mathcal{K}'(\varphi)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m(t_i)=\mathcal{E}_{2r}(\varphi)(t_i)h^{2r}+\mathcal{O}\big(\max\{h^{2r+2},\tilde{h}^2\}\big).\tag{4.4}$$

Note that

$$\big\|\mathcal{K}_m'(\varphi_m)\big\|\le\sup_{\substack{s,t\in[0,1]\\|u|\le\|\varphi\|_\infty+\epsilon}}|\kappa_u(s,t,u)|$$

and from [17, Proposition 4.2], we have

$$\Big\|\big(I-\mathcal{K}_m'(\varphi_m)\big)^{-1}\Big\|<\infty.$$

Thus, from (3.16) and (3.17), we have the following

$$\begin{aligned}&\Big\|\big(I-\mathcal{K}'(\varphi)\big)^{-1}\big[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\big]\big(I-\mathcal{K}_m'(\varphi_m)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\Big\|_\infty=\mathcal{O}\big(\tilde{h}^2\big),\\ &\Big\|\big(I-\mathcal{K}'(\varphi)\big)^{-1}\big[\mathcal{K}_m'(\varphi)-\mathcal{K}'(\varphi)\big]\big(I-\mathcal{K}_m'(\varphi_m)\big)^{-1}\mathcal{K}_m'(\varphi_m)(I-P_n)\varphi_m\Big\|_\infty=\mathcal{O}\big(\tilde{h}^2\big).\end{aligned}$$

Hence the required result follows from (4.3), (4.4) and the above two estimates. □

Before each of the following propositions, we prove lemmas and its corollaries which are used to prove next propositions.

Lemma 4.1. *Let P_n be the discrete orthogonal projection defined by* (3.8). *If $D^{(0,0,3)}\kappa\in C(\Omega)$ and $v\in C^{r+2}([0,1])$, then for $r\ge1$*

$$\mathcal{K}''(\varphi)(P_nv-v)^2=T(v)h^{2r}+\mathcal{O}(h^{2r+2}),\tag{4.5}$$

where

$$T(v)=\left(\int_0^1 J_r(\tau)^2 d\tau\right)\mathcal{K}''(\varphi)\left(v^{(r)}\right)^2.$$

Furthermore, when $r=1$, then

$$\mathcal{K}^{(3)}(\varphi)(P_n v-v)^3=\mathcal{O}\left(h^4\right). \tag{4.6}$$

Proof. The proofs of (4.5) and (4.6) follows from Lemma 3.2, [16, Lemma 2.4] and [16, Remark 2.4] respectively. □

Given that $(I-\mathcal{K}'(\varphi))^{-1}$ is a bounded linear operator. Let

$$\mathcal{M}=\left(I-\mathcal{K}'(\varphi)\right)^{-1}\mathcal{K}''(\varphi).$$

Note that $\mathcal{M}$ is a compact bi-linear integral operator. Also the smoothness of the kernel of $\mathcal{M}$ is the same as the kernel of $\mathcal{K}''(\varphi)$. See [6,28].

As a consequence of the above lemma, we get the following result.

Corollary 4.1. *For $r\geq 1$,*

$$\mathcal{M}(P_n v-v)^2=\mathcal{T}(v)h^{2r}+\mathcal{O}(h^{2r+2}), \tag{4.7}$$

where

$$\mathcal{T}(v)=\left(\int_0^1 J_r(\tau)^2 d\tau\right)\mathcal{M}\left(v^{(r)}\right)^2.$$

Lemma 4.2. *If $\varphi\in C^{r+2}([0,1])$, then for $r\geq 1$,*

$$\mathcal{K}''_m(\varphi_m)(z_n^G-\varphi_m)^2=T(\varphi)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right),$$

where T is defined in Lemma 4.1.

Proof. Note that

$$z_n^G-\varphi_m=P_n\left(z_n^S-\varphi_m\right)-(I-P_n)\varphi_m. \tag{4.8}$$

Thus,

$$\begin{aligned}\mathcal{K}''_m(\varphi_m)(z_n^G-\varphi_m)^2=&\mathcal{K}''_m(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right)\right)^2-2\mathcal{K}''_m(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right),(I-P_n)\varphi_m\right)\\&+\mathcal{K}''_m(\varphi_m)((I-P_n)\varphi_m)^2.\end{aligned} \tag{4.9}$$

Since $\|\mathcal{K}''_m(\varphi_m)\|<\infty$ and $\|P_n\|<\infty$, from (1.4b) it is easy to see that

$$\left\|\mathcal{K}''_m(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right)\right)^2\right\|_\infty=\mathcal{O}\left(\max\left\{h^{2r+4},\tilde{h}^4\right\}\right), \tag{4.10a}$$

$$\left\|\mathcal{K}''_m(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right),(I-P_n)\varphi_m\right)\right\|_\infty=\mathcal{O}\left(\tilde{h}^2\max\left\{h^{r+2},\tilde{h}^2\right\}\right). \tag{4.10b}$$

Now we write

$$\begin{aligned}
&\mathcal{K}_m''(\varphi_m)((I-P_n)\varphi_m)^2\\
=&\left[\mathcal{K}_m''(\varphi_m)-\mathcal{K}''(\varphi)\right]((I-P_n)\varphi_m)^2+\mathcal{K}''(\varphi)((I-P_n)\varphi_m)^2\\
=&\mathcal{K}''(\varphi)((I-P_n)\varphi)^2+\mathcal{K}''(\varphi)((I-P_n)(\varphi-\varphi_m))^2\\
&+\left[\mathcal{K}_m''(\varphi_m)-\mathcal{K}''(\varphi)\right]((I-P_n)\varphi_m)^2.
\end{aligned}$$

Since $\|\mathcal{K}_m''(\varphi_m)\|<\infty$ and $\|P_n\|<\infty$, from (3.14), (3.19) and the above estimate, we obtain

$$\mathcal{K}_m''(\varphi_m)((I-P_n)\varphi_m)^2=\mathcal{K}''(\varphi)((I-P_n)\varphi)^2+\mathcal{O}\left(\tilde{h}^2\right).$$

Therefore, from (4.5) we obtain

$$\mathcal{K}_m''(\varphi_m)((I-P_n)\varphi_m)^2=T(\varphi)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right).$$

Hence, the required result follows from (4.9), (4.10a), (4.10b) and the above equation. □

From the above lemma, we obtain

$$\left(I-\mathcal{K}'(\varphi)\right)^{-1}\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2=\mathcal{T}(\varphi)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right), \tag{4.11}$$

where $\mathcal{T}$ is defined by (4.7).

Lemma 4.3. *If* $\varphi\in C^{r+2}([0,1])$, *then*

$$\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3=\begin{cases}\mathcal{O}\left(\max\left\{h^4,\tilde{h}^2\right\}\right), & r=1,\\ \mathcal{O}\left(\max\left\{h^{3r},\tilde{h}^6\right\}\right), & r\geq 2.\end{cases}$$

Proof. First, let us consider the case when $r\geq 2$. Since

$$\left\|\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\right\|<\infty \quad \text{and} \quad \left\|\mathcal{K}_m^{(3)}(\varphi_m)\right\|<\infty,$$

from (1.4a) we obtain

$$\left\|\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3\right\|_\infty=\mathcal{O}\left(\max\left\{h^{3r},\tilde{h}^6\right\}\right).$$

Now consider the case, when $r=1$. We rewrite (4.8) as

$$(z_n^G-\varphi_m)^3=\left[P_n\left(z_n^S-\varphi_m\right)-(I-P_n)\varphi_m\right]^3.$$

Thus

$$\begin{aligned}
\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3=&\mathcal{K}_m^{(3)}(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right)\right)^3-\mathcal{K}_m^{(3)}(\varphi_m)((I-P_n)\varphi_m)^3\\
&-\mathcal{K}_m^{(3)}(\varphi_m)\left(\left(P_n\left(z_n^S-\varphi_m\right)\right)^2,(I-P_n)\varphi_m\right)\\
&+\mathcal{K}_m^{(3)}(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right),((I-P_n)\varphi_m)^2\right).
\end{aligned} \tag{4.12}$$

Since $\left\|\mathcal{K}_m^{(3)}(\varphi_m)\right\|<\infty$ and $\|P_n\|<\infty$, from (1.4b) and (3.15) we obtain

$$\begin{aligned}
&\mathcal{K}_m^{(3)}(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right)\right)^3=\mathcal{O}\left(h^6\right),\\
&\mathcal{K}_m^{(3)}(\varphi_m)\left(\left(P_n\left(z_n^S-\varphi_m\right)\right)^2,(I-P_n)\varphi_m\right)=\mathcal{O}\left(h^5\right),\\
&\mathcal{K}_m^{(3)}(\varphi_m)\left(P_n\left(z_n^S-\varphi_m\right),((I-P_n)\varphi_m)^2\right)=\mathcal{O}\left(h^4\right).
\end{aligned}$$

Note that, we have used the fact $\tilde{h}\leq h$ in the above three expressions. On the other hand, from (3.14), (3.20) we have

$$\mathcal{K}_m^{(3)}(\varphi_m)((I-P_n)\varphi_m)^3=\mathcal{K}^{(3)}(\varphi)((I-P_n)\varphi)^3+\mathcal{O}\left(\tilde{h}^2\right).$$

From (4.6), it follows that

$$\mathcal{K}_m^{(3)}(\varphi_m)((I-P_n)\varphi_m)^3=\mathcal{O}\left(\max\left\{h^4,\tilde{h}^2\right\}\right). \tag{4.13}$$

Now, combining the results (4.12) - (4.13), we obtain

$$\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3=\mathcal{O}\left(\max\left\{h^4,\tilde{h}^2\right\}\right).$$

Therefore

$$\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3=\mathcal{O}\left(\max\left\{h^4,\tilde{h}^2\right\}\right).$$

Hence follows the result. □

Proposition 4.2. *Let $\varphi\in C^{r+2}([0,1])$. Then for $r\geq 1$,*

$$\begin{aligned}
&\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right](s)\\
=&\frac{1}{2}\mathcal{T}(\varphi)(s)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right),
\end{aligned}$$

for all $s\in[0,1]$.

Proof. Applying the generalized Taylor's series expansion of $\mathcal{K}_m$ about φ_m in the neighbourhood $B(\varphi,\epsilon)$, we obtain

$$\begin{aligned}
&\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\\
=&\frac{1}{2}\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2+\frac{1}{6}\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3+\mathcal{R}_{4,m}\left(z_n^G-\varphi_m\right),
\end{aligned} \tag{4.14}$$

where

$$\mathcal{R}_{4,m}\left(z_n^G-\varphi_m\right)=\int_0^1\frac{(1-\theta)^3}{3!}\mathcal{K}_m^{(4)}\left(\varphi_m+\theta(z_n^G-\varphi_m)\right)(z_n^G-\varphi_m)^4\,d\theta.$$

Note that for any $x\in B(\varphi,\epsilon)$,

$$\mathcal{K}_m^{(4)}(x)v^4(s)=\frac{h}{p}\sum_{j=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\frac{\partial^4\kappa}{\partial u^4}\left(s,\mu_{q\nu}^j,x\left(\mu_{q\nu}^j\right)\right)v^4\left(\mu_{q\nu}^j\right),\quad v\in\mathcal{X},\quad s\in[0,1].$$

It follows that

$$\left\|\mathcal{K}_m^{(4)}(x)v^4\right\|_\infty\le\left(\sup_{\substack{s,t\in[0,1]\\|u|\le\|\varphi\|_\infty+\epsilon}}\left|\frac{\partial^4\kappa}{\partial u^4}(s,t,u)\right|\right)\|v\|_\infty^4.$$

Since φ_m and $z_n^G\in B(\varphi,\epsilon)$, $\varphi_m+\theta(z_n^G-\varphi_m)\in B(\varphi,\epsilon)$ and therefore

$$\left\|\mathcal{K}_m^{(4)}\left(\varphi_m+\theta(z_n^G-\varphi_m)\right)(z_n^G-\varphi_m)^4\right\|_\infty\le C\left\|z_n^G-\varphi_m\right\|_\infty^4=\mathcal{O}\left(h^{4r}\right).$$

It follows that

$$\mathcal{R}_{4,m}\left(z_n^G-\varphi_m\right)=\mathcal{O}\left(h^{4r}\right).$$

Using the resolvent identity (3.18) and (4.2), we obtain

$$\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2=\left[I-\mathcal{K}'(\varphi)\right]^{-1}\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2+\mathcal{O}\left(\tilde{h}^2\right).$$

By (4.11), it follows that

$$\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2=\mathcal{T}(\varphi)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right).$$

From the Lemma 4.3, we have

$$\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\mathcal{K}_m^{(3)}(\varphi_m)(z_n^G-\varphi_m)^3=\begin{cases}\mathcal{O}\left(\max\left\{h^4,\tilde{h}^2\right\}\right), & r=1,\\ \mathcal{O}\left(\max\left\{h^{3r},\tilde{h}^6\right\}\right), & r\ge 2.\end{cases}\tag{4.15}$$

Combining the results from (4.14) to (4.15), we obtain for $r\ge 1$,

$$\begin{aligned}&\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right]\\=&\frac{1}{2}\mathcal{T}(\varphi)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right),\end{aligned}$$

which completes the proof. □

Lemma 4.4. *If $v\in\mathcal{X}$, then for $r\ge 1$,*

$$\max_{0\le i\le n}\left|\mathcal{K}_m'(\varphi_m)(I-P_n)v(t_i)\right|\le C_6\|(I-P_n)v\|_\infty h^r,$$

where C_6 is a constant independent of h.

Proof. We write

$$\mathcal{K}_m'(\varphi_m)(I-P_n)v=\mathcal{K}_m'(\varphi)(I-P_n)v+[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)](I-P_n)v. \tag{4.16}$$

For fixed $s\in[0,1]$, let

$$\ell_{*,s}(t)=\ell_*(s,t)=\ell(s,t,\varphi(t))=\frac{\partial\kappa}{\partial u}(s,t,\varphi(t)),\quad t\in[0,1].$$

From the definition of $\mathcal{K}_m'(\varphi)$ and the discrete inner product, we have

$$\mathcal{K}_m'(\varphi)(I-P_n)v(s)=\sum_{j=1}^{n}\left\langle \ell_{*,s},(I-P_{n,j})v\right\rangle_{\Delta_j,m}.$$

Since $P_{n,j}$ is self-adjoint on $C(\Delta_j)$, so is $I-P_{n,j}$. Therefore

$$\mathcal{K}_m'(\varphi)(\mathcal{I}-P_n)v(s)=\sum_{j=1}^{n}\left\langle (I-P_{n,j})\ell_{*,s},(I-P_{n,j})v\right\rangle_{\Delta_j,m}.$$

Note that, if $s=t_i$ for some $i\in\{0,1,\cdots,n\}$, then $\ell_{*,s}\in C^r[t_{j-1},t_j]$ for all $j=1,\cdots,n$. Hence from (3.9),

$$\|(I-P_{n,j})\ell_{*,t_i}\|_{\Delta_j,\infty}\le C_1\left(\sup_{t\in[t_{j-1},t_j]}|D^{(0,r)}\ell_*(t_i,t)|\right)h^r.$$

Thus,

$$\begin{aligned}\max_{0\le i\le n}\left|\mathcal{K}_m'(\varphi)(I-P_n)v(t_i)\right|&\le\sum_{j=1}^{n}\left\|(I-P_{n,j})\ell_{m,s}\right\|_{\Delta_j,\infty}\|(I-P_{n,j})v\|_{\Delta_j,\infty}h\\&\le C_5\left\|(I-P_n)v\right\|_\infty h^r,\end{aligned}$$

where C_5 is a constant independent of h. Now, from (3.17), (4.16) and the above estimate, we obtain

$$\max_{0\le i\le n}\left|\mathcal{K}_m'(\varphi_m)(I-P_n)v(t_i)\right|\le C_6\left\|(I-P_n)v\right\|_\infty\max\left\{h^r,\tilde{h}^2\right\},$$

where $C_6=C_4+C_5$. Since $h^r\ge\tilde{h}^2$, the result follows. □

Recall that $\mathcal{L}_m=[I-\mathcal{K}_m'(\varphi_m)]^{-1}\mathcal{K}_m'(\varphi_m)$. Therefore, the proof of the following result is similar to that of the above lemma.

Corollary 4.2. *If $v\in\mathcal{X}$, then for $r\ge1$,*

$$\max_{0\le i\le n}|\mathcal{L}_m(I-P_n)v(t_i)|\le C_7\|(I-P_n)v\|_\infty h^r,$$

where C_7 is a constant independent of h.

Lemma 4.5. *Let $v\in\mathcal{X}$. If $r=1$, that is, when the range of P_n is the space of piecewise polynomials of degree zero, then*

$$\left\|(I-P_n)\mathcal{K}_m''(\varphi)(v,v)\right\|_\infty \le C_8 h\|v\|_\infty^2,$$

where C_8 is a constant independent of h.

Proof. Given that $\mathcal{X}_n$ is the space of piecewise constant functions with respect to the partition (3.2). Note that

$$\left\|(I-P_n)\mathcal{K}_m''(\varphi)(v,v)\right\|_\infty = \max_{1\le j\le n}\sup_{s\in[t_{j-1},t_j]}|(I-P_{n,j})\mathcal{K}_m''(\varphi)(v,v)(s)|.$$

Let $s\in[t_{j-1},t_j]$. Since the Legendre polynomial of degree zero $L_0(t)=1$ for all $t\in[0,1]$, we have from (3.11),

$$(I-P_n)\mathcal{K}_m''(\varphi)(v,v)(s)=\frac{1}{p}\sum_{\nu=1}^{p}\sum_{q=1}^{\rho}w_q\left[\mathcal{K}_m''(\varphi)(v,v)(s)-\mathcal{K}_m''(\varphi)(v,v)(t_{j-1}+\mu_{q\nu}h)\right]. \tag{4.17}$$

We also have

$$\begin{aligned}&\mathcal{K}_m''(\varphi)(v,v)(s)-\mathcal{K}_m''(\varphi)(v,v)(t_{j-1}+\mu_{q\nu}h)\\ =&\frac{h}{p}\sum_{k=1}^{n}\sum_{q=1}^{\rho}\sum_{\nu=1}^{p}w_q\left[\frac{\partial^2\kappa}{\partial u^2}\left(s,\mu_{q\nu}^k,\varphi\left(\mu_{q\nu}^k\right)\right)-\frac{\partial^2\kappa}{\partial u^2}\left(\mu_{q\nu}^j,\mu_{q\nu}^k,\varphi\left(\mu_{q\nu}^k\right)\right)\right]v^2\left(\mu_{q\nu}^k\right),\end{aligned}$$

where $\mu_{q\nu}^j=t_{j-1}+\mu_{q\nu}h$. For fixed $s\in[0,1]$, let

$$\lambda_{*,s}(t)=\lambda_*(s,t)=\lambda(s,t,\varphi(t))=\frac{\partial^2\kappa}{\partial u^2}(s,t,\varphi(t)),\quad t\in[0,1].$$

Then, by (3.5)

$$\begin{aligned}&\mathcal{K}_m''(\varphi)(v,v)(s)-\mathcal{K}_m''(\varphi)(v,v)(\mu_{q\nu}^j)=\sum_{k=1}^{n}\left\langle\lambda_{*,s}-\lambda_{*,\mu_{q\nu}^i},v^2\right\rangle_{\Delta_k,m}\\ =&\sum_{\substack{k=1\\k\neq j}}^{n}\left\langle\lambda_{*,s}-\lambda_{*,\mu_{q\nu}^j},v^2\right\rangle_{\Delta_k,m}+\left\langle\lambda_{*,s}-\lambda_{*,\mu_{q\nu}^j},v^2\right\rangle_{\Delta_j,m}.\end{aligned} \tag{4.18}$$

First consider the case when $k\neq j$. Applying Mean Value Theorem on the first component of $\lambda_*(\cdot,\cdot)$ in the interval $[s,\mu_{q\nu}^j]$, we obtain

$$\left\langle\lambda_{*,s}-\lambda_{*,\mu_{q\nu}^j},v^2\right\rangle_{\Delta_k,m}=(s-\mu_{q\nu}^j)\left\langle D^{(1,0)}\lambda_*(\theta_{q\nu}^j,\cdot),v^2\right\rangle_{\Delta_k,m},$$

for some $\theta^j_{qv}\in(t_{j-1},t_j)$, and the function $D^{(1,0)}\lambda_*(s,t)$, $t\in[t_{k-1},t_k]$ is given by

$$D^{(1,0)}\lambda_*(s,t)=\begin{cases} D^{(1,0)}\lambda_{1,*}(s,t)=\dfrac{\partial}{\partial s}\lambda_1(s,t,\varphi(t)), & 0\le t\le s\le 1,\\ D^{(1,0)}\lambda_{2,*}(s,t)=\dfrac{\partial}{\partial s}\lambda_2(s,t,\varphi(t)), & 0\le s\le t\le 1.\end{cases}$$

Therefore, for $k\neq j$,

$$\begin{aligned}\left|\left\langle \lambda_{*,s}-\lambda_{*,\mu^j_{qv}},v^2\right\rangle_{\Delta_k,m}\right| &\le \left|s-\mu^j_{qv}\right|\left(\sup_{s\neq t}\left|D^{(1,0)}\lambda_*(s,t)\right|\right)\|v\|^2_\infty h\\ &\le\left(\sup_{s\neq t}\left|D^{(1,0)}\lambda_*(s,t)\right|\right)\|v\|^2_\infty h^2,\end{aligned}$$

where

$$\begin{aligned}\sup_{s\neq t}|D^{(1,0)}\lambda_*(s,t)| &=\max\left\{\sup_{0\le t<s\le 1}|D^{(1,0)}\lambda_{1,*}(s,t)|,\ \sup_{0\le s<t\le 1}|D^{(1,0)}\lambda_{2,*}(s,t)|\right\}\\ &=\max\left\{\sup_{\substack{0\le t<s\le 1\\ |u|\le\|\varphi\|_\infty}}|D^{(1,0,2)}\kappa_1(s,t,u)|,\ \sup_{\substack{0\le s<t\le 1\\ |u|\le\|\varphi\|_\infty}}|D^{(1,0,2)}\kappa_2(s,t,u)|\right\}.\end{aligned}$$

On the other hand,

$$\left|\left\langle \lambda_{*,s}-\lambda_{*,\mu^j_{qv}},v^2\right\rangle_{\Delta_j,m}\right|\le 2\left(\sup_{0\le s,t\le 1}|\lambda_*(s,t)|\right)\|v\|^2_\infty h,$$

where

$$\sup_{0\le s,t\le 1}|\lambda_*(s,t)|=\sup_{\substack{0\le s,t\le 1\\ |u|\le\|\varphi\|_\infty}}|D^{(0,0,2)}\kappa(s,t,u)|.$$

Then, from (4.18) we obtain

$$\begin{aligned}&|\mathcal{K}''_m(\varphi)(v,v)(s)-\mathcal{K}''_m(\varphi)(v,v)(\mu^j_{qv})|\\ \le&\left(\sum_{\substack{k=1\\k\neq j}}^n\left(\sup_{s\neq t}|D^{(1,0)}\lambda_*(s,t)|\right)\|v\|^2_\infty h^2\right)+2\left(\sup_{0\le s,t\le 1}|\lambda_*(s,t)|\right)\|v\|^2_\infty h.\end{aligned}$$

It follows that

$$|\mathcal{K}''_m(\varphi)(v,v)(s)-\mathcal{K}''_m(\varphi)(v,v)(\mu^j_{qv})|\le C_8\|v\|^2h,$$

where

$$C_8=\left(\sup_{s\neq t}|D^{(1,0)}\lambda_*(s,t)|\right)+2\left(\sup_{0\leq s,t\leq 1}|\lambda_*(s,t)|\right).$$

The result now follows from (4.17) and the above estimate. □

Corollary 4.3. *Let $v\in\mathcal{X}$. If $r=1$, that is, when the range of P_n is the space of piecewise polynomials of degree zero, then*

$$\left\|(I-P_n)\mathcal{K}_m'(\varphi)v\right\|_\infty\leq C_9 h\|v\|_\infty,$$

where C_9 is a constant independent of h.

Proof. The proof is similar to that of Lemma 4.5. □

Proposition 4.3. *Let t_i be any point of the partition $\Delta^{(n)}$ defined by (3.2). Then*

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right](t_i)\\&=\begin{cases}\mathcal{O}\left(h^4\right), & r=1,\\ \mathcal{O}\left(\max\{h^{3r},h^r\tilde{h}^4\}\right), & r\geq 2.\end{cases}\end{aligned}$$

Proof. Generalized Taylor's series expansion gives

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right]\\&=\frac{1}{2}\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2+\mathcal{L}_m(I-P_n)\mathcal{R}_{3,m}\left(z_n^G-\varphi_m\right),\end{aligned}\tag{4.19}$$

where

$$\mathcal{R}_{3,m}\left(z_n^G-\varphi_m\right)=\int_0^1\frac{(1-\theta)^2}{2!}\mathcal{K}_m^{(3)}\left(\varphi_m+\theta(z_n^G-\varphi_m)\right)(z_n^G-\varphi_m)^3\,d\theta.$$

It follows that

$$\left\|\mathcal{R}_{3,m}\left(z_n^G-\varphi_m\right)\right\|_\infty\leq\frac{1}{6}\left(\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\epsilon}}\left|\frac{\partial^3\kappa}{\partial u^3}(s,t,u)\right|\right)\left\|z_n^G-\varphi_m\right\|_\infty^3.$$

Therefore, by (1.4a)

$$\left\|\mathcal{R}_{3,m}\left(z_n^G-\varphi_m\right)\right\|_\infty=\mathcal{O}\left(\max\{h^{3r},\tilde{h}^6\}\right).$$

Since $\|I-P_n\|\leq 1+\|P_n\|<\infty$, from Corollary 4.2, it is easy to see that

$$\mathcal{L}_m(I-P_n)\mathcal{R}_{3,m}\left(z_n^G-\varphi_m\right)(t_i)=\mathcal{O}\left(\max\left\{h^{4r},h^r\tilde{h}^6\right\}\right).\tag{4.20}$$

First consider the case $r\geq 2$. Since $\|\mathcal{K}_m''(\varphi_m)\|<\infty$ and $\|I-P_n\|_\infty<\infty$, by (1.4a) and the Corollary 4.2, we have

$$\frac{1}{2}\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2(t_i)=\mathcal{O}\left(\max\left\{h^{3r},h^r\tilde{h}^4\right\}\right),\quad r\geq 2.$$

When $r=1$, we write

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2\\=&\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m''(\varphi_m)-\mathcal{K}_m''(\varphi)\right](z_n^G-\varphi_m)^2+\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2.\end{aligned}$$

By (1.4a), (3.14) and the Lemma 3.3, we have

$$\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m''(\varphi_m)-\mathcal{K}_m''(\varphi)\right](z_n^G-\varphi_m)^2=\mathcal{O}\left(h^4\right).$$

On the other hand

$$\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2(t_i)=\sum_{j=1}^n\left\langle \ell_{m,t_i},(I-P_{n,j})\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2\right\rangle_{\Delta_j,m}.$$

Since $I-P_{n,j}$ is self-adjoint,

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2(t_i)\\=&\sum_{j=1}^n\left\langle (I-P_{n,j})\ell_{m,t_i},(I-P_{n,j})\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2\right\rangle_{\Delta_j,m}.\end{aligned}$$

It follows that

$$\begin{aligned}&\max_{0\leq i\leq n}\left|\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2(t_i)\right|\\\leq&\sum_{j=1}^n\left\|(I-P_{n,j})\ell_{m,t_i}\right\|_{\Delta_j,\infty}\left\|(I-P_{n,j})\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2\right\|_{\Delta_j,\infty}h.\end{aligned}$$

By Corollary 4.2 and Lemma 4.5, we obtain

$$\max_{0\leq i\leq n}|\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi)(z_n^G-\varphi_m)^2(t_i)|=\mathcal{O}\left(h^4\right),\quad r=1.$$

Therefore

$$\frac{1}{2}\mathcal{L}_m(I-P_n)\mathcal{K}_m''(\varphi_m)(z_n^G-\varphi_m)^2(t_i)=\begin{cases}\mathcal{O}\left(h^4\right), & r=1,\\ \mathcal{O}\left(\max\left\{h^{3r},h^r\tilde{h}^4\right\}\right), & r\geq 2.\end{cases}\tag{4.21}$$

Then combing (4.19), (4.20) and (4.21), we obtain

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right](t_i)\\=&\begin{cases}\mathcal{O}\left(h^4\right), & r=1,\\ \mathcal{O}\left(\max\left\{h^{3r},h^r\tilde{h}^4\right\}\right), & r\geq 2.\end{cases}\end{aligned}$$

This follows the result. □

We quote the following result from [18, Proposition 1, Proposition 6], which will be used in the next proposition.

$$\left\|(I-P_n)\mathcal{K}_m'(\varphi)(I-P_n)\varphi\right\|_\infty=\begin{cases}\mathcal{O}\left(h^3\right), & r=1,\\ \mathcal{O}\left(h^{r+2}\right), & r\geq 2.\end{cases} \tag{4.22}$$

Proposition 4.4. *If φ_m and z_n^G are respectively the Nyström and the discrete Galerkin approximation of φ, then*

$$\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)(t_i)=\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right).$$

Proof. Adding and subtracting $\mathcal{K}_m'(\varphi)$, we have

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\\ =&\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\right](z_n^G-\varphi_m)+\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(z_n^G-\varphi_m).\end{aligned}$$

Then, using (1.4a), (3.17) and the Corollary 4.2, we obtain for $r\geq 1$,

$$\begin{aligned}&\left|\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m'(\varphi_m)-\mathcal{K}_m'(\varphi)\right](z_n^G-\varphi_m)(t_i)\right|\\ \leq& C_7(1+\|P_n\|)h^r\tilde{h}^2\left(\max\left\{h^r,\tilde{h}^2\right\}\right).\end{aligned} \tag{4.23}$$

Note that

$$z_n^G-\varphi_m=P_nz_n^S-\varphi_m=P_n\left(z_n^S-\varphi\right)-(I-P_n)\varphi+(\varphi-\varphi_m).$$

Then

$$\begin{aligned}&\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(z_n^G-\varphi_m)\\ =&\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)P_n\left(z_n^S-\varphi\right)-\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(I-P_n)\varphi\\ &+\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(\varphi-\varphi_m).\end{aligned}$$

By the Corollary 4.2

$$\left|\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)P_n\left(z_n^S-\varphi\right)(t_i)\right|\leq C\left\|(I-P_n)\mathcal{K}_m'(\varphi)P_n\left(z_n^S-\varphi\right)\right\|h^r,$$

then by (1.4b) and Corollary 4.3, we obtain

$$\left|\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)P_n\left(z_n^S-\varphi\right)(t_i)\right|=\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right)\quad\text{for}\quad r\geq 1.$$

Also, the Corollary 4.2 and (4.22) implies

$$\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(I-P_n)\varphi(t_i)=\mathcal{O}\left(h^{2r+2}\right)\quad\text{for}\quad r\geq 1.$$

It is easy to see (from (3.14) and Corollary 4.2) that

$$\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(\varphi-\varphi_m)=\mathcal{O}\left(h^r\tilde{h}^2\right) \quad \text{for} \quad r\geq 1.$$

Therefore, for $r\geq 1$,

$$\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi)(z_n^G-\varphi_m)(t_i)=\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right).$$

Hence, the result follows from (4.23) and the above equation. □

We prove the main theorem as follows.

Theorem 4.1. *Let $\mathcal{K}$ be the Urysohn integral operator with Green's function type kernel κ defined by* (1.2), *and let $\mathcal{K}_m$ be the Nyström approximation of it defined by* (3.13). *Let φ be the unique solution of the equation (1.1). Assume that* 1 *is not an eigenvalue of $\mathcal{K}'(\varphi)$. Let $\mathcal{X}_n$ be the space of piecewise polynomials of degree $\leq r-1$ with respect to the partition $\Delta^{(n)}:=0=t_0<t_1<\cdots<t_n=1$ defined by* (3.2). *Let $P_n:L^\infty[0,1]\to\mathcal{X}_n$ be the discrete orthogonal projection defined by* (3.8) *and $\tilde{z}_n^S$ be the discrete iterated Galerkin approximation of φ. Then*

$$\left(z_n^S-\varphi\right)(t_i)=\left[\mathcal{E}_{2r}(\varphi)(t_i)+\frac{1}{2}\mathcal{T}(\varphi)(t_i)\right]h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right),$$

where the operators $\mathcal{E}_{2r}$ and $\mathcal{T}$ are respectively defined by (3.12) *and* (4.5).

Proof. We have from (4.1)

$$\begin{aligned} z_n^S-\varphi=&\left[I-\mathcal{K}_m'(\varphi_m)\right]^{-1}\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right]\\ &-\mathcal{L}_m(I-P_n)\left[\mathcal{K}_m(z_n^G)-\mathcal{K}_m(\varphi_m)-\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)\right]\\ &-\mathcal{L}_m(I-P_n)\mathcal{K}_m'(\varphi_m)(z_n^G-\varphi_m)-\mathcal{L}_m(I-P_n)\varphi_m+\varphi_m-\varphi. \end{aligned}$$

The result now follows from (3.14), Propositions 4.1-4.4. □

We now apply Richardson extrapolation to obtain an approximation of φ with higher order of convergence. Define

$$z_n^{EX}=\frac{2^{4r}z_{2n}^S-z_n^S}{2^{4r}-1}.$$

We choose the partitions $\Delta^{(m)}$ and $\Delta^{(n)}$ such that $m^2\geq n^{2r+2}$. Then, it is easy to see from the Theorem 4.1, that

$$\left(z_n^{EX}-\varphi\right)(t_i)=\mathcal{O}\left(h^{2r+2}\right) \quad \text{for all} \quad i=1,2,\cdots,n. \tag{4.24}$$

5 Numerical results

For the numerical results, we consider the Example 5.1 with $a=2$, $b=\frac{2}{3}$ and $c=\sqrt{12}$ and Example 5.2 from the subsection 2.1.

Example 5.1. For

$$\varphi(s)-\int_0^1 \kappa(s,t)[\psi(t,\varphi(t))]dt=f(s), \quad 0\le s\le 1, \tag{5.1}$$

where

$$\kappa(s,t)=\frac{1}{c\sinh(c)}\begin{cases} \sinh(cs)\sinh(c(1-t)), & 0\le t\le s\le 1,\\ c(1-s)\sinh(ct), & 0\le s\le t\le 1,\end{cases}$$

and

$$\psi(t,\varphi(t))=c^2\varphi(t)-2(\varphi(t))^3, \quad t\in[0,1].$$

We have

$$f(s)=\frac{1}{\sinh(c)}\left\{2\sinh(c(1-s))+\frac{2}{3}\sinh(cs)\right\}.$$

The exact solution of (5.1) is given by

$$\varphi(s)=\frac{2}{2s+1}, \quad s\in[0,1].$$

Example 5.2. For

$$x(s)=\int_0^1 k(s,t)[g(t,x(t))+f(t)]dt, \tag{5.2}$$

where

$$k(s,t)=\begin{cases} t(s-1), & 0\le t\le s,\\ s(t-1), & s\le t\le 1,\end{cases}$$

where

$$g(t,u)=\frac{1}{1+t+u}$$

and the function f is so chosen such that

$$\varphi(s)=\frac{s(1-s)}{1+s}$$

is a solution of (5.2).

Let $\mathcal{X}_n$ be the space of piecewise constant functions with respect to the uniform partition $\Delta^{(n)}$ of the interval $[0,1]$. Let $P_n: L^\infty[0,1]\to\mathcal{X}_n$ be the discrete orthogonal projection defined by (3.7)-(3.8).

Let $t_i=\frac{i-1}{20}$, $i=1,2,\cdots,21$ be the partition points with step size $h=\frac{1}{20}$. The numerical quadrature is chosen to be the composite 2 point Gaussian quadrature rule with respect

	Example 5.1 :		$\varphi(s)=\frac{2}{2s+1}$		**Example 5.2 :**		$\varphi(s)=\frac{s(1-s)}{1+s}$	
t_i	$\epsilon_n^S(t_i):n=20$	δ^S	$\epsilon_n^{EX}(t_i):n=20$	δ^{EX}	$\epsilon_n^S(t_i):n=20$	δ^S	$\epsilon_n^{EX}(t_i):n=20$	δ^{EX}
0.05	8.6×10^{-3}	2.00	2.98×10^{-6}	3.99	1.84×10^{-3}	2.001	5.79×10^{-7}	4.003
0.1	7.56×10^{-3}	2.00	2.23×10^{-6}	3.99	1.63×10^{-3}	2.001	4.38×10^{-7}	4.003
0.15	6.79×10^{-3}	2.00	1.59×10^{-6}	3.99	1.46×10^{-3}	2.001	3.37×10^{-7}	4.003
0.2	6.22×10^{-3}	2.00	1.09×10^{-6}	3.97	1.31×10^{-3}	2.000	2.62×10^{-7}	4.004
0.25	5.78×10^{-3}	2.00	7.13×10^{-7}	3.96	1.19×10^{-3}	2.000	2.07×10^{-7}	4.004
0.3	5.45×10^{-3}	2.00	4.46×10^{-7}	3.94	1.08×10^{-3}	2.000	1.64×10^{-7}	4.004
0.35	5.19×10^{-3}	2.00	2.7×10^{-7}	3.91	9.93×10^{-4}	2.000	1.32×10^{-7}	4.005
0.4	4.98×10^{-3}	2.00	1.69×10^{-7}	3.86	9.13×10^{-4}	2.000	1.07×10^{-7}	4.005
0.45	4.82×10^{-3}	2.00	1.3×10^{-7}	3.83	8.43×10^{-4}	2.000	8.69×10^{-8}	4.006
0.5	4.68×10^{-3}	2.00	1.41×10^{-7}	3.85	7.82×10^{-4}	2.000	7.14×10^{-8}	4.006
0.55	4.55×10^{-3}	2.00	1.91×10^{-7}	3.89	7.27×10^{-4}	2.000	5.93×10^{-8}	4.007
0.6	4.44×10^{-3}	2.00	2.72×10^{-7}	3.93	6.78×10^{-4}	2.000	4.97×10^{-8}	4.007
0.65	4.33×10^{-3}	2.00	3.75×10^{-7}	3.95	6.35×10^{-4}	2.000	4.22×10^{-8}	4.007
0.7	4.22×10^{-3}	2.00	4.95×10^{-7}	3.97	5.95×10^{-4}	2.000	3.64×10^{-8}	4.007
0.75	4.10×10^{-3}	2.00	6.26×10^{-7}	3.98	5.59×10^{-4}	2.000	3.19×10^{-8}	4.007
0.8	3.98×10^{-3}	2.00	7.6×10^{-7}	3.99	5.26×10^{-4}	2.000	2.84×10^{-8}	4.006
0.85	3.84×10^{-3}	2.00	8.94×10^{-7}	3.99	4.95×10^{-4}	2.000	2.58×10^{-8}	4.005
0.9	3.69×10^{-3}	2.00	1.02×10^{-6}	3.99	4.67×10^{-4}	2.000	2.40×10^{-8}	4.004
0.95	3.52×10^{-3}	2.00	1.14×10^{-6}	4	4.41×10^{-4}	2.000	2.27×10^{-8}	4.002

to partition $\Delta^{(m)}$ with $m=n^2$ subintervals. Then $\tilde{h}=h^2$. Therefore, it is expected from the Theorem 4.1 and Eq. (4.24), that

$$\epsilon_n^S(t_i)=|\varphi(t_i)-z_n^S(t_i)|=\mathcal{O}\left(h^2\right) \quad \text{and} \quad \epsilon_n^{EX}(t_i)=|\varphi(t_i)-z_n^{EX}(t_i)|=\mathcal{O}\left(h^4\right),$$

where

$$z_n^{EX}(t_i)=\frac{4z_{2n}^S(t_i)-z_n^S(t_i)}{3}.$$

Let δ^S and δ^{EX} be respectively the orders of convergence of z_n^S and z_n^{EX} at the partition points. We expect $\delta^S=2$ and $\delta^{EX}=4$.

From the table above, it is clear that the obtained orders of convergence match well with the theoretical orders of convergence. Also the order of convergence of the extrapolated solution improves upon the discrete iterated Galerkin solution.

6 Conclusions

Iterated Galerkin method is applied to (1.1) and the following asymptotic series expansion is proved in Rakshit et al [26].

$$\varphi_n^S(t_i)-\varphi(t_i)=\zeta_{2r}(t_i)h^{2r}+\mathcal{O}\left(h^{2r+2}\right), \tag{6.1}$$

where φ_n^S is the iterated Galerkin solution and ζ_{2r} is a function bounded by a constant independent of h. Then using Richardson extrapolation and the above equation, an approximate solution with order convergence h^{2r+2} can be obtained.

In this paper, we have proved a similar result for the discrete iterated Galerkin solution z_n^S. We consider the case of discrete orthogonal projection P_n instead of the orthogonal projection π_n, and the Nyström approximations instead of the integral operators.

Euler-McLaurin series expansion plays an important role in the proof of (6.1). The main difficulty here is that there is no discrete version of the Euler-McLaurin series. So, we approach in a different way to prove a discrete version of (6.1), and obtain the following result.

$$z_n^S(t_i)=\varphi(t_i)+\gamma_{2r}(t_i)h^{2r}+\mathcal{O}\left(\max\left\{h^{2r+2},\tilde{h}^2\right\}\right), \tag{6.2}$$

where the function γ_{2r} is independent of h. Note that we chose a fine partition (with m subintervals) that defines the composite quadrature rule and a coarse partition ($\Delta^{(n)}$) with n subintervals to define the approximating space $\mathcal{X}_n$ ($m=np$, $p\in\mathbb{N}$). If we choose the composite numerical integration formula associated with the coarse partition $\Delta^{(n)}$, the error in the Nyström approximations would be of the order h^2 even for a higher order quadrature rule, as the kernel of $\mathcal{K}$ is of the type of Green's function. In that case, z_n^S would be of the order h^2 due to this discretization. Hence we needed to choose a different partition for the quadrature rule which makes the proofs more involved. It is to be noted that even if $m>n$, the size of the system of equations that need to be solved for the above approximations remains nr ($=\dim(\mathcal{X}_n)$).

It will be of interest to have an asymptotic error expansion (like (6.2)) for the discrete iterated modified projection solution (see in [18]).

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