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Time-Space Adaptive Finite Element Method for Nonlinear Schrödinger Equation

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Abstract. This paper is devoted to adaptive finite element method for the nonlinear Schrödinger equation. The adaptive method is based on the extrapolation technology and a second order accurate, linear and mass preserving finite element scheme. For error control, we take the difference between the numerical gradient and the recovered gradient obtained by the superconvergent cluster recovery method as the spatial discretization error estimator and the difference of numerical approximations between two consecutive time steps as the temporal discretization error estimator. A time-space adaptive algorithm is developed for numerical approximation of the nonlinear Schrödinger equation. Numerical experiments are presented to illustrate the reliability and efficiency of the proposed error estimators and the corresponding adaptive algorithm.

AMS subject classifications: 65N15, 65N30, 65N50

Key words: Nonlinear Schrödinger equation, finite element method, error estimators, time-space adaptive algorithm.

1 Introduction

In this paper, we focus on adaptive finite element method for the following nonlinear Schrödinger equation

$$iu_t+\alpha\Delta u+(v(x)+\beta|u|^2)u=0 \quad \text{in } \Omega\times(0,T], \tag{1.1}$$

subject to the initial condition

$$u(x,0)=u_0(x) \quad \text{in } \Omega\times\{0\}, \tag{1.2}$$

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and satisfies the homogeneous Dirichlet boundary condition, where Ω is a bounded open domain in R^2, $v(x)$,u denote the real-valued potential energy function and complex-valued wave function, respectively, α is the given parameter as well as β. And $i=\sqrt{-1}$ is the imaginary unit, Δ is the Laplace operator, $|\cdot|$ is the module of a complex number. It is well known that one of the important properties for equation (1.1) is the mass conservation. Computing the inner product of (1.1) with u, and taking the real parts, one can obtain the following mass conservative identity

$$\frac{d}{dt}\int_{\Omega}|u|^2dx=0. \tag{1.3}$$

The nonlinear Schrödinger equation, which was originated from quantum mechanics, is one of the most important equations in mathematical physics. It has been widely used to model various nonlinear physical phenomena, such as underwater acoustics [24], nonlinear optics [20,22], quantum condensates [11] and other nonlinear phenomena [1].

The Schrödinger equation is the most basic equation in quantum mechanics, and researchers cannot prove it from any more fundamental assumptions. Its correctness can only be tested by practice, and it is not easy to obtain the exact solution for the complex Schrödinger equation. Therefore, it is necessary to study the numerical solution of Schrödinger equation. The numerical approximations of the Schrödinger equation have been extensively investigated in the past few decades. As for the spatial discretization, it mainly includes the finite difference method [2,3,6,12], the spectral method [5], the finite element method [4,7,23,25], the discontinuous finite element method [10,19], the hybrid finite element method [18], the two grid mesh method [14,26] and references therein. And for the temporal discretization, there are Crank-Nicolson scheme [8], scalar variable auxiliary (SAV) method [9] and so on. The blow-up property of solutions is one of the important properties for the Schrödinger equation. Merle and Tsutsumi [21] proved that, for a blow-up solution with a radially symmetric initial data, the origin is a blow-up point and an L_2-concentration phenomenon occurs at the origin. In order to simulate the blow-up phenomenon described by the nonlinear Schrödinger equation, it is natural to use adaptive mesh techniques so the fine meshes are only used in a small neighborhood of the blow-up solution. The subject of a posteriori error estimates and adaptive methods for the nonlinear Schrödinger equation has been studied by some authors, we refer to [15–17] and the references therein. In [15, 17], a posteriori error estimates for Crank-Nicolson finite element schemes of linear evolution Schrödinger equations were considered. T. Katsaounis and I. Kyzaa derived an optimal order a posteriori error estimates in the $L^\infty(L^2)$-norm for relaxation time discrete and fully discrete schemes of the nonlinear Schrödinger equation up to the critical exponent in [16].

In this work, we shall be interested in the numerical approximation of the blow-up solution of (1.1) with some radially symmetric initial value (1.2) in two dimensions, which is proposed by using a time-space adaptive method for the nonlinear Schrödinger equation. The adaptive method is based on the extrapolation technology and a second order accurate, linear and mass preserving finite element scheme. For the error control, we