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The Exponential Transformation Based Lattice Boltzmann Model for Convection-Diffusion Equation

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Abstract. In this paper, an exponential transformation based lattice Boltzmann (LB) model for solving the n-dimensional (nD) convection-diffusion equation(CDE) is developed. Firstly, a class of exponential transformation is proposed to convert the nD CDE into a diffusion equation. Then, the converted diffusion equation is solved by the LB model. So, compared to the available LB models for CDE, the present LB model can eliminate the difficulty in treating the convection term. With the direct Taylor expansion method, it is shown that the CDE can be exactly derived from the exponential transformation based LB model. Finally, a variety of numerical tests have been conducted to validate the present LB model. It can be found that the numerical results agree well with the analytical solutions. Moreover, we also find that the present LB model has second-order convergence rate in space, and it is more effective and more stable than the previous LB model for the CDE.

AMS subject classifications: 76M28, 65N75

Key words: Lattice Boltzmann model, convection-diffusion equation, exponential transformation.

1 Introduction

As an important kind of partial differential equations (PDEs), the convection-diffusion equation (CDE) has gained much attention in the study of complex phenomena in many

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fields. However, it is usually difficult to obtain the analytical solutions of most of the CDEs. With the development of computing technique, some numerical approaches have been developed to solve the CDE, including the explicit and implicit finite-difference method, finite-element method, finite-volume method, meshless method, and so on [1–8].

The mesoscopic lattice Boltzmann method (LBM) originated from kinetic theory [9–13], has gained much attention in solving nonlinear systems. As an alternative to the traditional numerical methods, the LBM has many advantages for its distinct features, such as the simplicity of programming, locality of computation, and ease in dealing with complex boundaries. In recent years, the LBM has also been successfully applied to solve the Poisson equation, Laplace equation, Burgers' equation, reaction-diffusion equation, CDE, and so on [14–25].

Although many LB models have been proposed for the CDE, there are still some problems. The first is that when the simple linear equilibrium distribution function is used, we need to add an additional source term in the evolution equation to eliminate the effect of the convection term. The second is that if a more complex equilibrium distribution with the quadratic form is applied, a numerical diffusion would be induced. The third is that the convection term usually causes a numerical instability problem in the LBM, and it should be treated properly. Thus, if the convection term in the CDE can be eliminated, the problems inherited in these LB models for the CDE will be overcome.

As early as 1962, Brenner et al. [26] first adopted a transformation to reduce the CDE to a heat conduction equation. In the year of 2013, Zhang et al. [27] introduced a similar exponential transformation to eliminate the convection term in the nonlinear delay convection-reaction-diffusion equations. However, only one-dimensional CDEs were solved in their works. In this paper, the exponential transformation is extended to convert the n-dimensional (nD) CDE into a nD diffusion equation, then a LB method is developed to solve the converted nD diffusion equation. In the exponential transformation based LBM, we can use the simple linear equilibrium distribution function and discrete velocity lattice model, and thus the computation efficiency and stability of the LBM for CDE can be improved.

The rest of the paper is organized as follows. In Section 2, an exponential transformation based LB model for nD CDE is proposed. In Section 3, some numerical simulations are performed to test the present LB model, and finally some conclusions are given in Section 4.

2 Lattice Boltzmann model for convection-diffusion equation

In this section, the nD CDE is first converted into a diffusion equation by the exponential transformation, and then a LB model for the converted diffusion equation is developed.

In this paper, the following nD CDE with a source term is considered,

$$\partial_t \rho + \mathbf{u}\cdot\nabla\rho = \nabla\cdot(\alpha\nabla\rho) + F(\mathbf{x},t), \tag{2.1}$$