Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 1, pp. 263-294

DOI: 10.4208/aamm.OA-2023-0025
February 2025

Error Analysis of a New Euler Semi-Implicit Time-Discrete Scheme for the Incompressible MHD System with Variable Density

Yuan Li* and Xuewei Cui

College of Mathematics and Physics, Wenzhou University, Wenzhou, Zhejiang 325035, China

Received 16 January 2023; Accepted (in revised version) 9 March 2023

Abstract. The incompressible magnetohydrodynamics system with variable density is coupled by the incompressible Navier-Stokes equations with variable density and the Maxwell equations. In this paper, we study a new first-order Euler semi-discrete scheme for solving this system. The proposed numerical scheme is unconditionally stable for any time step size $\tau > 0$. Furthermore, a rigorous error analysis is presented and the first-order temporal convergence rate $\mathcal{O}(\tau)$ is derived by using the method of mathematical induction and the discrete maximal L^p-regularity of the Stokes problem. Finally, numerical results are given to support the theoretical analysis.

AMS subject classifications: 65N30, 76M05

Key words: Magnetohydrodynamics, variable density flows, Euler semi-implicit scheme, error analysis.

1 Introduction

In this paper, we consider the 3D incompressible magnetohydrodynamics (MHD) system with variable density. It is used to describe the motions of several conducting incompressible immiscible fluids without surface tension in presence of a magnetic field, and has a wide range of applications in physical and industrial fields, such as astrophysics, geophysics, plasma physics and liquid metals of an aluminum electrolysis cell (cf. [8, 16, 27, 39]). This MHD system is governed by the following nonlinear parabolic system in $Q_T = \Omega \times (0,T]$:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1a}$$

*Corresponding author.
Email: liyuan@wzu.edu.cn (Y. Li)

$$\rho(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})-\frac{1}{Re}\Delta\mathbf{u}+S\mathbf{b}\times\mathbf{curl}\ \mathbf{b}+\nabla p=\mathbf{f}, \tag{1.1b}$$

$$\mathbf{b}_t+\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b})-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=0, \tag{1.1c}$$

$$\nabla\cdot\mathbf{u}=0,\quad \nabla\cdot\mathbf{b}=0, \tag{1.1d}$$

where $\Omega\subset\mathbf{R}^3$ is a bounded and convex domain with the boundary $\Gamma=\partial\Omega$ and $[0,T]$ is the time interval with some $T>0$. The vector function $\mathbf{f}$ is a given body force. Positive constants Re, Rm, S represent the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The unknown functions are the density ρ, the velocity field $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$.

The MHD system (1.1a)-(1.1d) is supplemented with the following initial values and boundary conditions:

$$\begin{cases} \rho(x,0)=\rho_0(x), & \mathbf{u}(x,0)=\mathbf{u}_0(x), & \mathbf{b}(x,0)=\mathbf{b}_0(x) & \text{in } \Omega, \\ \mathbf{u}(x,t)=\mathbf{g}(x,t), & \mathbf{b}(x,t)\cdot\mathbf{n}=0, & \mathbf{curl}\ \mathbf{b}(x,t)\times\mathbf{n}=0 & \text{on } \Sigma_T, \\ \rho(x,t)=a(x,t) & & & \text{on } \Sigma_T^{in}, \end{cases} \tag{1.2}$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary Γ, $\Sigma_T=\Gamma\times(0,T]$, $\Sigma_T^{in}=\Gamma_{in}\times(0,T]$ and Γ_{in} is the inflow boundary defined by $\Gamma_{in}=\{x\in\Gamma:\ \mathbf{g}\cdot\mathbf{n}<0\}$. For the reason of simplicity, we consider the homogeneous Dirichlet boundary condition for the velocity field, i.e., $\mathbf{g}=0$. This means that the boundary is impermeable, i.e., $\Gamma_{in}=\emptyset$.

In addition, we require that the initial velocity field $\mathbf{u}_0$ and magnetic field $\mathbf{b}_0$ satisfy the incompressible conditions, i.e.,

$$\nabla\cdot\mathbf{u}_0=\nabla\cdot\mathbf{b}_0=0 \tag{1.3}$$

and there is no vacuum state in Ω, i.e.,

$$0<\min_{x\in\overline{\Omega}}\rho_0(x):=m\le\rho_0(x)\le M:=\max_{x\in\overline{\Omega}}\rho_0(x). \tag{1.4}$$

There are many studies on the well-posedness of the solutions to the MHD system (1.1a)-(1.2). The global existence of weak solutions of finite energy in the whole space $\mathbf{R}^3$ was firstly established by Gerbeau and Le Bris under no vacuum assumption (1.4) in [15]. The global existence of strong solutions with small initial data in some Besov space was proved by Abidi and Paicu in [1]. Chen, Tan and Wang in [11] considered the local strong solutions in the presence of vacuum. Huang and Wang in [26] extended it to the global existence of strong solutions for 2D problem. Other theoretical results can be found in [7,12,23,34] and references cited therein.

On numerical methods for the incompressible MHD problems, there have a large amount of works for the constant density MHD problem, such as [5,6,14,24,33,37,38,40,42–46] and references cited therein. However, few studies are made for the variable density MHD problem since the equations in (1.1a)-(1.1d) entangle hyperbolic, parabolic and elliptic features.

We recall some known numerical methods for the incompressible Navier-Stokes flows with variable density. The first-order and second-order Gauge-Uzawa projection schemes were proposed by Pyo and Shen in [41] and the first-order temporal convergence rate was shown by Chen, Mao and Shen for the first-order Gauge-Uzawa scheme in [10]. An incremental projection scheme was proposed by Guermond and Quartapelle in [20]. However, the scheme in [20] is somewhat expensive since there have two-consuming projection steps. Moreover, the pressure equation is a variable-coefficient elliptic equation. To avoid the variable-coefficient elliptic equation, Guermond and Salgado in [21] proposed a fractional-step splitting scheme based upon a pressure Poisson equation, and the error estimate was given in [22] where the authors assumed that the numerical solution of density has the upper and lower boundness uniformly. By using the similar assumption, An studied a new fractional-step scheme, an iteration penalty scheme and the Euler semi-implicit scheme proposed in [35], and proved the first-order temporal convergence rate in [2–4], respectively. Recently, a complete error estimate of a linearized finite element scheme for the two-dimensional Navier-Stokes equations with variable density was studied in [9], where the numerical error was split into the temporal error and spatial error. The temporal error was estimated by using the discrete maximal L^p-regularity of the nonlinear parabolic equations developed in [28], and the spatial error was estimated by using the energy techniques developed in [29–32]. The complete error estimates for the three-dimensional problem were studied in [30], where the discontinuous Galerkin method was used to discretize the density equation.

As mentioned above, few work was studied for numerical simulations of the MHD system with variable density. Gerbeau, Le Bris and Lelièvre in [16] discussed the numerical approximations in the ALE formulation for the two-fluid MHD equations with variable density, where the ALE method was used to track the interface of two fluids. An Euler semi-implicit finite element scheme based on weak ALE formulation was considered and proved to be stable in [16]. By the implicit-explicit linearized technique, a first-order Euler semi-discrete scheme was proposed in [36], where the first-order temporal convergence rate was derived.

Let X_h, $\mathbf{V}_h$, M_h and $\mathbf{W}_h$ for approximations of the density, the velocity field, the pressure and the magnetic field, respectively. Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of the time interval $[0,T]$ with time step $\tau=T/N$ and $t_n=n\tau$ for $0\le n\le N$. Denote $\mathbf{u}_h^n\in\mathbf{V}_h$ and $\rho_h^n\in X_h$ the finite element approximations of $\mathbf{u}(t_n)$ and $\rho(t_n)$, respectively. To preserve the unconditional stability of the first-order Euler finite element scheme, the following equality is often used:

$$
\begin{aligned}
&\left(\rho_h^n\frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\tau},\mathbf{u}_h^{n+1}\right)+\frac{1}{2}\left(\frac{\rho_h^{n+1}-\rho_h^n}{\tau},|\mathbf{u}_h^{n+1}|^2\right)\\
=&\frac{1}{2\tau}\left(\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2-\|\sigma_h^n\mathbf{u}_h^{n+1}\|_{L^2}^2+\|\sigma_h^n(\mathbf{u}_h^{n+1}-\mathbf{u}_h^n)\|_{L^2}^2\right), \qquad (1.5)
\end{aligned}
$$

where $\sigma_h^n=\sqrt{\rho_h^n}$. This can be viewed as the discrete version of the following continuous equality on the time derivative of the kinetic energy:

$$(\rho\mathbf{u}_t,\mathbf{u})+\frac{1}{2}(\rho_t,|\mathbf{u}|^2)=\frac{1}{2}\frac{d}{dt}\|\sigma\mathbf{u}\|_{L^2}^2,$$

where $\sigma=\sqrt{\rho}$. Thus, to preserve the unconditional stability of the numerical scheme, one requires $\mathbf{u}_h^{n+1}\cdot\mathbf{u}_h^{n+1}\in X_h$ and the constraint condition of finite element spaces:

$$\mathbf{V}_h\cdot\mathbf{V}_h\subset X_h. \tag{1.6}$$

The same constraint condition (1.6) is needed for finite element schemes of the incompressible Navier-Stokes equations with variable density (cf. [9]) and with mass diffusion (cf. [17–19]).

A natural question is whether an unconditionally stable algorithm can be constructed without the constraint condition (1.6). This paper gives a positive answer. Firstly, we rewrite the MHD system (1.1a)-(1.1d) to an equivalent MHD system (see (2.5a)-(2.5d) below). Then we propose a time-discrete scheme for the new MHD system by the implicit-explicit linearized technique. In terms of the discrete maximal L^p-regularity in [9,28], the first-order convergence rate $\mathcal{O}(\tau)$ is shown for the density, the velocity field, the pressure and the magnetic field in some norms. In the finite element discretizations, we use (P_1,P_1b,P_1,P_1) finite element discretization for the density, the velocity field, the pressure and the magnetic field, respectively. The unconditional stability of the fully discrete scheme is proved. Thus, the constraint condition (1.6) is avoided.

This paper is organized as follows. In Section 2, we recall some notations and give the equivalent MHD system to (1.1a)-(1.1d). Then, we propose a time-discrete scheme for the new system and prove the unconditional stability of algorithm. The main result about the first-order convergence rate is presented in Section 3. The proof is given by the method of mathematical induction and the discrete maximal L^p-regularity for the Stokes problem in Section 4. In Section 5, numerical results are shown to confirm the temporal convergence rate by using the finite element method to discrete the spatial direction. Moreover, the unconditional stability of the fully discrete scheme is presented.

2 Time-discrete algorithm

2.1 Preliminaries

For the mathematical setting, we introduce the following notations. For $k\in\mathbb{N}^+$ and $1\le p\le+\infty$, the boldface notations $\mathbf{H}^k(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector-value Sobolev spaces corresponding to $H^k(\Omega)^3$, $W^{k,p}(\Omega)^3$ and $L^p(\Omega)^3$, respectively. Moreover, the norms $\|\cdot\|_{H^k}$, $\|\cdot\|_{W^{k,p}}$ and $\|\cdot\|_{L^p}$ in $\mathbf{H}^k(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are defined by the classical way. Denote $\mathbf{H}_0^k(\Omega)$ be the subspace of $\mathbf{H}^k(\Omega)$ where the functions have zero trace on $\partial\Omega$. We use $(\cdot,\cdot)$ to denote the L^2 or $\mathbf{L}^2$ inner product. In the rest of this paper, we use the symbol C to denote some positive constant which can depend on the physical data and is independent of the time step size τ.

Introduce the following function spaces:

$$\begin{aligned}
&\mathbf{V}=\mathbf{H}_0^1(\Omega), \quad \mathbf{W}=\{\mathbf{w}\in\mathbf{H}^1(\Omega), \mathbf{w}\cdot\mathbf{n}=0 \text{ on } \Gamma\}, \quad \mathbf{W}_0=\{\mathbf{w}\in\mathbf{W}, \nabla\cdot\mathbf{w}=0\},\\
&\mathbf{L}_\sigma^q(\Omega)=\{\mathbf{v}\in\mathbf{L}^q(\Omega), \nabla\cdot\mathbf{v}=0, \mathbf{v}\cdot\mathbf{n}|_\Gamma=0\}, \quad \mathbf{L}_\sigma^q(\Omega)^\perp=\{\nabla\phi, \phi\in W^{1,q}(\Omega)\},\\
&\mathbf{V}_\sigma=\{\mathbf{v}\in\mathbf{V}, \nabla\cdot\mathbf{v}=0\}, \quad M=L_0^2(\Omega)=\left\{q\in L^2(\Omega), \int_\Omega q dx=0\right\}.
\end{aligned}$$

The norm in $\mathbf{W}$ can be defined by

$$\|\mathbf{w}\|_W=\left(\int_\Omega(|\mathbf{curl}\ \mathbf{w}|^2+|\nabla\cdot\mathbf{w}|^2)dx\right)^{1/2}, \quad \forall\mathbf{w}\in\mathbf{W}.$$

As we know that there exists some $C>0$ such that

$$\|\mathbf{w}\|_{H^1}\leq C\|\mathbf{w}\|_W, \quad \forall\mathbf{w}\in\mathbf{W},$$

which implies that

$$\|\mathbf{w}\|_{H^1}\leq C\|\mathbf{curl}\ \mathbf{w}\|_{L^2}, \quad \forall\mathbf{w}\in\mathbf{W}_0.$$

By the Poincaré inequality, the norms in $\mathbf{V}$ is defined by

$$\|\mathbf{v}\|_V=\|\nabla\mathbf{v}\|_{L^2}=\left(\int_\Omega|\nabla\mathbf{v}|^2dx\right)^{1/2}, \quad \forall\mathbf{v}\in\mathbf{V}.$$

Furthermore, there holds

$$\|\mathbf{v}\|_{H^1}\leq C\|\nabla\mathbf{v}\|_{L^2}, \quad \forall\mathbf{v}\in\mathbf{V}.$$

For $2\leq q<+\infty$, we introduce the projection operator $\mathbb{P}:\mathbf{L}^q(\Omega)\to\mathbf{L}_\sigma^q(\Omega)$. In terms of the Helmholtz-Weyl decomposition of $\mathbf{L}^q(\Omega)$ (cf. [13]), for any function $\mathbf{v}\in\mathbf{L}^q(\Omega)$, there has a unique decomposition

$$\mathbf{v}=\mathbb{P}\mathbf{v}+\nabla\phi \tag{2.1}$$

with $\nabla\phi\in\mathbf{L}_\sigma^q(\Omega)^\perp$ such that

$$\|\mathbb{P}\mathbf{v}\|_{L^q}+\|\nabla\phi\|_{L^q}\leq C\|\mathbf{v}\|_{L^q}. \tag{2.2}$$

Finally, we define the Stokes operator $\mathbf{A}$ by $\mathbf{A}=-\mathbb{P}\Delta$ which satisfies

$$\|\mathbf{v}\|_{W^{2,q}}\leq C\|\mathbf{A}\mathbf{v}\|_{L^q}, \quad \forall\mathbf{v}\in\mathbf{V}_\sigma\cap\mathbf{W}^{2,q}(\Omega). \tag{2.3}$$

2.2 An equivalent system

It is easy to prove that the solution $(\rho,\mathbf{u},\mathbf{b})$ to (1.1a)-(1.2) satisfies the following energy inequalities (cf. [36]):

$$\sup_{t\in[0,T]}\left\{\frac{1}{2}\|\sigma(t)\mathbf{u}(t)\|_{L^2}^2+\frac{S}{2}\|\mathbf{b}(t)\|_{L^2}^2+\int_0^t\left(\frac{1}{2Re}\|\nabla\mathbf{u}(s)\|_{L^2}^2+\frac{S}{Rm}\|\mathbf{curl}\ \mathbf{b}(s)\|_{L^2}^2\right)ds\right\}$$
$$\leq\frac{1}{2}\|\mathbf{u}_0\|_{L^2}^2+\frac{S}{2}\|\mathbf{b}_0\|_{L^2}^2+\frac{Re}{2}\int_0^T\|\mathbf{f}(t)\|_{L^2}^2dt, \tag{2.4}$$

where $\sigma(t)=\sqrt{\rho(t)}$. We note that the quantity $\frac{1}{2}\|\sigma(t)\mathbf{u}(t)\|_{L^2}^2$ is the kinetic energy of the flow, which gives a hint to rewrite the system (1.1a)-(1.1d) as an equivalent form.

From (1.1a) and $\nabla\cdot\mathbf{u}=0$, by a simple calculation, we get

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=2\sigma\left(\frac{\partial\sigma}{\partial t}+\nabla\sigma\cdot\mathbf{u}\right)=2\sigma\left(\frac{\partial\sigma}{\partial t}+\nabla\cdot(\sigma\mathbf{u})\right)=0$$

and

$$\rho\frac{\partial\mathbf{u}}{\partial t}=\sigma\frac{\partial(\sigma\mathbf{u})}{\partial t}+\frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{u}).$$

Then we can rewrite the MHD system (1.1a)-(1.1d) to the following equivalent form:

$$\sigma_t+\nabla\cdot(\sigma\mathbf{u})=0, \tag{2.5a}$$
$$\sigma(\sigma\mathbf{u})_t-\frac{1}{Re}\Delta\mathbf{u}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{u})+S\mathbf{b}\times\mathbf{curl}\ \mathbf{b}+\nabla p=\mathbf{f}, \tag{2.5b}$$
$$\mathbf{b}_t+\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b})-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=0, \tag{2.5c}$$
$$\nabla\cdot\mathbf{u}=0,\quad \nabla\cdot\mathbf{b}=0. \tag{2.5d}$$

From (1.2), initial and boundary conditions for (2.5a)-(2.5d) are

$$\begin{cases}\sigma(x,0)=\sigma_0(x)=\sqrt{\rho_0(x)}, & \mathbf{u}(x,0)=\mathbf{u}_0(x), & \mathbf{b}(x,0)=\mathbf{b}_0(x) & \text{in }\ \Omega,\\ \mathbf{u}(x,t)=0, & \mathbf{b}(x,t)\cdot\mathbf{n}=0, & \mathbf{curl}\ \mathbf{b}(x,t)\times\mathbf{n}=0 & \text{on }\ \Sigma_T,\end{cases} \tag{2.6}$$

where σ_0 satisfies the no vacuum assumption:

$$\sqrt{m}\leq\sigma_0(x)\leq\sqrt{M},\quad \forall x\in\Omega. \tag{2.7}$$

2.3 Time-discrete scheme

Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform partition of the time interval $[0,T]$ with time step size $\tau=T/N$ and $t_n=n\tau$. For any sequence of functions $\{g^n\}_{n=0}^N$, we denote

$$D_\tau g^n=\frac{g^n-g^{n-1}}{\tau}\quad \text{for}\ \ 1\leq n\leq N.$$

Start with $\sigma^0=\sigma_0$, $\mathbf{u}^0=\mathbf{u}_0$ and $\mathbf{b}^0=\mathbf{b}_0$. For $0\leq n\leq N-1$, we propose the following Euler time-discrete scheme for solving (2.5a)-(2.6) numerically:

Algorithm 2.1 First-order Euler time-discrete algorithm.

1. Step I: For given σ^n and $\mathbf{u}^n$, we first solve σ^{n+1} by

$$D_\tau \sigma^{n+1} + \nabla\cdot(\sigma^{n+1}\mathbf{u}^n) = 0. \tag{2.8}$$

2. Step II: Then we solve $(\mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1})$ by

$$\begin{aligned}&\sigma^{n+1} D_\tau(\sigma^{n+1}\mathbf{u}^{n+1}) - \frac{1}{Re}\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} + \rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} \\ &\quad + \frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n) + S(\mathbf{b}^n \times \mathbf{curl}\ \mathbf{b}^{n+1}) = \mathbf{f}^{n+1}, \quad \nabla\cdot\mathbf{u}^{n+1} = 0,\end{aligned} \tag{2.9}$$

and

$$D_\tau \mathbf{b}^{n+1} + \frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b}^{n+1}) - \mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n) = 0, \quad \nabla\cdot\mathbf{b}^{n+1} = 0, \tag{2.10}$$

with boundary conditions $\mathbf{u}^{n+1} = 0$, $\mathbf{b}^{n+1}\cdot\mathbf{n} = 0$ and $\mathbf{curl}\ \mathbf{b}^{n+1}\times\mathbf{n} = 0$ on Γ, where $\rho^{n+1} = (\sigma^{n+1})^2$.

2.4 The stability of the algorithm

In this subsection, we will prove the unconditional stability of the proposed time-discrete algorithm (2.8)-(2.10). Since the sub-problems are linear problems by using the implicit-explicit linearized method, the unconditional stability implies the existence and uniqueness of numerical solutions $(\sigma^{n+1}, \mathbf{u}^{n+1}, p^{n+1}, \mathbf{b}^{n+1})$.

Multiplying (2.8) by $2\tau\sigma^{n+1}$ and integrating over Ω, we have

$$\|\sigma^{n+1}\|_{L^2}^2 + \|\sigma^{n+1} - \sigma^n\|_{L^2}^2 - \|\sigma^n\|_{L^2}^2 = 0, \tag{2.11}$$

where we noted that

$$2(\nabla\cdot(\sigma^{n+1}\mathbf{u}^n), \sigma^{n+1}) = 2(\nabla\sigma^{n+1}\cdot\mathbf{u}^n, \sigma^{n+1}) = (\nabla|\sigma^{n+1}|^2, \mathbf{u}^n) = -(|\sigma^{n+1}|^2, \nabla\cdot\mathbf{u}^n) = 0$$

by using $\nabla\cdot\mathbf{u}^n = 0$. Then from (2.11) we get the L^2 stability of σ^{n+1}:

$$\|\sigma^{n+1}\|_{L^2} \le \|\sigma_0\|_{L^2}, \quad \forall 0 \le n \le N-1.$$

Testing (2.9) by $\mathbf{v} = 2\tau\mathbf{u}^{n+1}$ leads to

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 + \|\sigma^{n+1}\mathbf{u}^{n+1} - \sigma^n\mathbf{u}^n\|_{L^2}^2 - \|\sigma^n\mathbf{u}^n\|_{L^2}^2 + \frac{2\tau}{Re}\|\nabla\mathbf{u}^{n+1}\|_{L^2} \\ &\quad + 2\tau S(\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1}, \mathbf{u}^{n+1}) = 2\tau(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}),\end{aligned} \tag{2.12}$$

where we use

$$\begin{aligned}&2(\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1},\mathbf{u}^{n+1})\\=&(\rho^{n+1}\mathbf{u}^n,\nabla|\mathbf{u}^{n+1}|^2)\\=&-(\mathbf{u}^{n+1}\nabla\cdot(\rho^{n+1}\mathbf{u}^n),\mathbf{u}^{n+1}).\end{aligned}$$

Testing (2.10) by $\mathbf{w}=2S\tau\mathbf{b}^{n+1}$, we get

$$\begin{aligned}&S\left(\|\mathbf{b}^{n+1}\|_{L^2}^2+\|\mathbf{b}^{n+1}-\mathbf{b}^n\|_{L^2}^2-\|\mathbf{b}^n\|_{L^2}^2\right)+\frac{2S\tau}{Rm}\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}\\&-2S\tau(\mathbf{u}^{n+1}\times\mathbf{b}^n,\mathbf{curl}\ \mathbf{b}^{n+1})=0.\end{aligned}\tag{2.13}$$

According to the vector formula

$$(\mathbf{a}\times\mathbf{curl}\ \mathbf{b},\mathbf{c})=(\mathbf{c}\times\mathbf{a},\mathbf{curl}\ \mathbf{b}),\quad\forall\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{R}^3,\tag{2.14}$$

we have

$$(\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1},\mathbf{u}^{n+1})=(\mathbf{u}^{n+1}\times\mathbf{b}^n,\mathbf{curl}\ \mathbf{b}^{n+1}).$$

Then adding (2.12) and (2.13) and summing up from $n=0$ to $n=m$ with $0\le m\le N-1$, we have

$$\begin{aligned}&\|\sigma^{m+1}\mathbf{u}^{m+1}\|_{L^2}^2+S\|\mathbf{b}^{m+1}\|_{L^2}^2+\sum_{n=0}^{m}(\|\sigma^{n+1}\mathbf{u}^{n+1}-\sigma^n\mathbf{u}^n\|_{L^2}^2+\|\mathbf{b}^{n+1}-\mathbf{b}^n\|_{L^2}^2)\\&\quad+\tau\sum_{n=0}^{m}\left(\frac{1}{Re}\|\nabla\mathbf{u}^{n+1}\|_{L^2}+\frac{2S}{Rm}\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}\right)\\&\le\|\sigma_0\mathbf{u}_0\|_{L^2}^2+S\|\mathbf{b}_0\|_{L^2}^2+C\tau\sum_{n=0}^{N-1}\|\mathbf{f}^{n+1}\|_{L^2}.\end{aligned}\tag{2.15}$$

3 Main result

In this section, we will present the main result on the first-order temporal convergence rate $\mathcal{O}(\tau)$ of numerical solutions $(\sigma^{n+1},\mathbf{u}^{n+1},p^{n+1},\mathbf{b}^{n+1})$ in different norms, and then prove these convergence rates.

For $0\le n\le N$, we introduce error functions as follows:

$$\begin{aligned}&e_\rho^n=\rho(t_n)-\rho^n,\quad e_\sigma^n=\sigma(t_n)-\sigma^n,\quad \mathbf{e}_\mathbf{u}^n=\mathbf{u}(t_n)-\mathbf{u}^n,\\&e_p^n=p(t_n)-p^n,\quad \mathbf{e}_\mathbf{b}^n=\mathbf{b}(t_n)-\mathbf{b}^n.\end{aligned}$$

Since we focus on the optimal convergence analysis for the time discrete scheme (2.8)-(2.10), the regularity assumption of the exact solution is needed. Throughout this paper, we assume that the equivalent MHD system (2.5a)-(2.6) have a unique strong solution which is sufficiently smooth such that

$$\sigma\in L^\infty(0,T;H^3(\Omega)),\quad \sigma_t\in L^\infty(0,T;H^1(\Omega)), \tag{3.1a}$$

$$\sigma_{tt}\in L^\infty(0,T;W^{1,\infty}(\Omega))\cap L^2(0,T;H^2(\Omega)), \tag{3.1b}$$

$$\mathbf{u}\in\mathbf{L}^\infty(0,T;\mathbf{V}\cap\mathbf{W}^{2,4}(\Omega)),\quad \mathbf{u}_{tt}\in\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)), \tag{3.1c}$$

$$\mathbf{u}_t\in\mathbf{L}^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega))\cap\mathbf{L}^2(0,T;\mathbf{H}^2(\Omega)), \tag{3.1d}$$

$$\mathbf{b}\in\mathbf{L}^\infty(0,T;\mathbf{W}\cap\mathbf{W}^{2,4}(\Omega)),\quad \mathbf{b}_t\in\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)),\quad \mathbf{b}_{tt}\in\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega)). \tag{3.1e}$$

The main result in this paper is presented in the following theorem.

Theorem 3.1. *Suppose that the initial data $\sigma_0\in H^2(\Omega)\cap W^{1,\infty}(\Omega)$, $\boldsymbol{u}_0\in\boldsymbol{W}^{1,\infty}(\Omega)\cap\boldsymbol{V}_\sigma\cap\boldsymbol{H}^2(\Omega)$, $\boldsymbol{b}_0\in\boldsymbol{W}_0$, $\boldsymbol{f}\in L^2(0,T;\boldsymbol{L}^2(\Omega))$, and σ_0 satisfies the no vacuum assumption* (2.7). *Let σ, $\boldsymbol{u}$, p and $\boldsymbol{b}$ be the solution of the equivalent MHD system* (2.5a)-(2.6) *and satisfy the regularity assumption* (3.1). *Then for $0\le n\le N-1$, the time-discrete scheme* (2.8)-(2.10) *admit the unique solution σ^{n+1}, $\boldsymbol{u}^{n+1}$, p^{n+1} and $\boldsymbol{b}^{n+1}$. Furthermore, there exists some small $\tau_0>0$ such that when $\tau<\tau_0$, one has*

$$\sqrt{m}\le\sigma^k(x)\le\sqrt{M},\quad \forall x\in\Omega,\quad 0\le k\le N, \tag{3.2}$$

and the regularity estimates

$$\max_{0\le k\le N}\Big(\|\boldsymbol{u}^k\|_{W^{1,\infty}}+\|\boldsymbol{A}\boldsymbol{u}^k\|_{L^2}+\|\sigma^k\|_{W^{1,\infty}}+\|\sigma^k\|_{H^2}\Big)\le K, \tag{3.3}$$

where

$$\begin{aligned}K=&\|\boldsymbol{u}\|_{L^\infty(0,T;W^{1,\infty})}+\|\boldsymbol{u}\|_{L^\infty(0,T;H^2)}+\|\sigma\|_{L^\infty(0,T;W^{1,\infty})}+\|\sigma\|_{L^\infty(0,T;H^2)}\\&+\|\boldsymbol{u}_0\|_{W^{1,\infty}}+\|\boldsymbol{A}\boldsymbol{u}_0\|_{L^2}+\|\sigma_0\|_{W^{1,\infty}}+\|\sigma_0\|_{H^2}+1.\end{aligned}$$

Moreover, the following temporal error estimates hold:

$$\max_{1\le n\le N}\Big(\|e_\rho^n\|_{L^2}+\|e_\sigma^n\|_{L^2}+\|\nabla\boldsymbol{e}_{\boldsymbol{u}}^n\|_{L^2}+\|\boldsymbol{curl}\ \boldsymbol{e}_{\boldsymbol{b}}^n\|_{L^2}\Big)\le C\tau, \tag{3.4a}$$

$$\tau\sum_{n=1}^N\Big(\|\boldsymbol{e}_{\boldsymbol{u}}^n\|_{H^2}^2+\|\boldsymbol{e}_{\boldsymbol{b}}^n\|_{H^2}^2+\|e_p^n\|_{H^1}^2\Big)\le C\tau^2. \tag{3.4b}$$

To prove Theorem 3.1, we recall three lemmas which are frequently used. The following two lemmas established in [9] play a key role in error analysis.

Lemma 3.1. *Assume that $\sigma^n\in H^2(\Omega)\cap W^{1,\infty}(\Omega)$ and $\boldsymbol{u}^n\in\boldsymbol{W}^{1,\infty}(\Omega)\cap\boldsymbol{V}_\sigma\cap\boldsymbol{H}^2(\Omega)$ are given for $0\le n\le N-1$. Then the hyperbolic equation* (2.8) *has a unique solution $\sigma^{n+1}\in H^2(\Omega)\cap L^\infty(\Omega)$ which satisfies the maximum principle*

$$\min_{x\in\overline{\Omega}}\sigma^n(x)\le\sigma^{n+1}(x)\le\max_{x\in\overline{\Omega}}\sigma^n(x),\quad \forall\ x\in\Omega. \tag{3.5}$$

Lemma 3.2. *For $1\le n\le N$, let $a^n(x)$ be a function defined on Ω which satisfies*

(1) $\kappa_0\le a^n(x)\le\kappa_1$ *for some positive constants κ_0 and κ_1,*

(2) $\max_{1\le n\le N}\|a^n\|_{W^{1,4}}\le\kappa_2$ *for some positive constant κ_2,*

(3) $\sum_{n=1}^{N}\|a^n-a^{n-1}\|_{L^\infty}\le\kappa_3$ *for some positive constant κ_3.*

Then the solution $\boldsymbol{v}^n$ to the following problem

$$a^n D_\tau \boldsymbol{v}^n+\boldsymbol{A}\boldsymbol{v}^n=\boldsymbol{g}^n,\quad n=1,\cdots,N,$$

satisfies

$$\tau\sum_{n=1}^{N}\left(\|D_\tau\boldsymbol{v}^n\|_{L^q}^p+\|\boldsymbol{A}\boldsymbol{v}^n\|_{L^q}^p\right)\le C\left(\tau^{1-p}\|\boldsymbol{v}^0\|_{L^q}+\tau\|\boldsymbol{A}\boldsymbol{v}^0\|_{L^q}\right)+C\tau\sum_{n=1}^{N}\|\boldsymbol{g}^n\|_{L^q}^p, \tag{3.6}$$

where $C>0$ is independent of τ and a^n, but may depends on κ_0, κ_1, κ_2, κ_3 and T.

Then, we recall the discrete Gronwall's inequality established in [25].

Lemma 3.3. *Let a_k,b_k and γ_k be the nonnegative numbers such that*

$$a_n+\tau\sum_{k=0}^{n}b_k\le\tau\sum_{k=0}^{n}\gamma_k a_k+B\quad \textit{for } n\ge 1. \tag{3.7}$$

Suppose $\tau\gamma_k<1$ and set $\sigma_k=(1-\tau\gamma_k)^{-1}$. Then there holds:

$$a_n+\tau\sum_{k=0}^{n}b_k\le\exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right)B\quad \textit{for}\ \ n\ge 1. \tag{3.8}$$

Remark 3.1. If the sum on the right-hand side of (3.7) extends only up to $n-1$, then the estimate (3.8) still holds for all $k\ge 1$ with $\sigma_k=1$.

Next, we give error equations as follows. For $0\le n\le N-1$, the exact solution $(\sigma,\mathbf{u},p,\mathbf{b})$ to (2.5a)-(2.6) at $t=t_{n+1}$ satisfies

$$D_\tau\sigma(t_{n+1})+\nabla\cdot(\sigma(t_{n+1})\mathbf{u}(t_n))=R_\sigma^{n+1}, \tag{3.9}$$

and

$$\begin{aligned}&\sigma(t_{n+1})D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))-\frac{1}{Re}\Delta\mathbf{u}(t_{n+1})+\rho(t_{n+1})(\mathbf{u}(t_n)\cdot\nabla)\mathbf{u}(t_{n+1})\\&\quad+\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(\rho(t_{n+1})\mathbf{u}(t_n))+\nabla p(t_{n+1})+S(\mathbf{b}(t_n)\times\mathbf{curl}\ \mathbf{b}(t_{n+1}))=\mathbf{f}^{n+1}+\mathbf{R}_{\mathbf{u}}^{n+1}\end{aligned} \tag{3.10}$$

with $\nabla\cdot\mathbf{u}(t_{n+1})=0$, and

$$D_\tau\mathbf{b}(t_{n+1})+\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b}(t_{n+1}))-\mathbf{curl}\ (\mathbf{u}(t_{n+1})\times\mathbf{b}(t_n))=\mathbf{R}_{\mathbf{b}}^{n+1} \tag{3.11}$$

with $\nabla\cdot\mathbf{b}(t_{n+1})=0$, where truncation error functions R_σ^{n+1}, $\mathbf{R}_{\mathbf{u}}^{n+1}$ and $\mathbf{R}_{\mathbf{b}}^{n+1}$ are given by

$$\begin{aligned}
R_\sigma^{n+1}=&(D_\tau\sigma(t_{n+1})-\sigma_t(t_{n+1}))-\nabla\sigma(t_{n+1})\cdot(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)),\\
\mathbf{R}_{\mathbf{u}}^{n+1}=&\sigma(t_{n+1})(D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))-(\sigma\mathbf{u})_t(t_{n+1}))\\
&-\rho(t_{n+1})((\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\cdot\nabla)\mathbf{u}(t_{n+1})-\frac{\mathbf{u}(t_{n+1})}{2}\nabla\rho(t_{n+1})\cdot(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\\
&-S((\mathbf{b}(t_{n+1})-\mathbf{b}(t_n))\times\mathbf{curl}\ \mathbf{b}(t_{n+1})),\\
\mathbf{R}_{\mathbf{b}}^{n+1}=&(D_\tau\mathbf{b}(t_{n+1})-\mathbf{b}_t(t_{n+1}))+\mathbf{curl}\ (\mathbf{u}(t_{n+1})\times(\mathbf{b}(t_{n+1})-\mathbf{b}(t_n))).
\end{aligned}$$

In terms of the regularity assumption (3.1) and the Taylor formula, truncation error functions satisfy

$$\|R_\sigma^{n+1}\|_{W^{1,\infty}}^2+\tau\sum_{n=0}^{N-1}\Big(\|R_\sigma^{n+1}\|_{H^2}^2+\|\mathbf{R}_{\mathbf{u}}^{n+1}\|_{H^1}^2+\|\mathbf{R}_{\mathbf{b}}^{n+1}\|_{L^2}^2\Big)\le C\tau^2. \tag{3.12}$$

Subtracting (2.8), (2.9), (2.10) from (3.9), (3.10), (3.11), we get the following error equations:

$$D_\tau e_\sigma^{n+1}+\nabla\cdot(\sigma(t_{n+1})\mathbf{e}_{\mathbf{u}}^n)+\nabla\cdot(e_\sigma^{n+1}\mathbf{u}^n)=R_\sigma^{n+1}, \tag{3.13a}$$

$$\nabla\cdot\mathbf{e}_{\mathbf{u}}^{n+1}=0,\quad \sigma^{n+1}D_\tau(\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1})-\frac{1}{Re}\Delta\mathbf{e}_{\mathbf{u}}^{n+1}+\nabla e_p^{n+1}+\sum_{i=1}^{4}I_i^{n+1}=\mathbf{R}_{\mathbf{u}}^{n+1}, \tag{3.13b}$$

$$\nabla\cdot\mathbf{e}_{\mathbf{b}}^{n+1}=0,\quad D_\tau\mathbf{e}_{\mathbf{b}}^{n+1}+\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1})-I_5^{n+1}=\mathbf{R}_{\mathbf{b}}^{n+1}, \tag{3.13c}$$

where $I_1^{n+1},\cdots,I_5^{n+1}$ are given by

$$\begin{aligned}
I_1^{n+1}=&e_\rho^{n+1}(\mathbf{u}(t_n)\cdot\nabla)\mathbf{u}(t_{n+1})+\rho^{n+1}(\mathbf{e}_{\mathbf{u}}^n\cdot\nabla)\mathbf{u}(t_{n+1})+\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{e}_{\mathbf{u}}^{n+1},\\
I_2^{n+1}=&e_\sigma^{n+1}D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1})+\sigma^{n+1}e_\sigma^{n+1}D_\tau\mathbf{u}(t_{n+1})\\
&+\sigma^{n+1}(R_\sigma^{n+1}-\nabla(\sigma(t_{n+1}))\cdot\mathbf{e}_{\mathbf{u}}^n-\nabla e_\sigma^{n+1}\cdot\mathbf{u}^n)\mathbf{u}(t_n),\\
I_3^{n+1}=&\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(e_\rho^{n+1}\mathbf{u}(t_n))+\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(\rho^{n+1}\mathbf{e}_{\mathbf{u}}^n)+\frac{\mathbf{e}_{\mathbf{u}}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n),\\
I_4^{n+1}=&S\mathbf{e}_{\mathbf{b}}^n\times\mathbf{curl}\ \mathbf{b}(t_{n+1})+S\mathbf{b}^n\times\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1},\\
I_5^{n+1}=&\mathbf{curl}\ (\mathbf{u}(t_{n+1})\times\mathbf{e}_{\mathbf{b}}^n)+\mathbf{curl}\ (\mathbf{e}_{\mathbf{u}}^{n+1}\times\mathbf{b}^n).
\end{aligned}$$

Proof of Theorem 3.1. We will use the method of mathematical induction to prove the maximal principle (3.2) and the regularity estimate (3.3). Meanwhile, error estimates (3.4a)-(3.4b) can be derived in the procedure of induction.

It is clear that (3.2) and (3.3) hold for $k=0$. For $0\le n\le N-1$, we assume that (3.2) and (3.3) are valid for $k=n$, i.e., there hold

$$\sqrt{m}\le\sigma^n(x)\le\sqrt{M},\quad \forall x\in\Omega, \tag{3.14}$$

and

$$\|\mathbf{u}^n\|_{W^{1,\infty}}+\|\mathbf{u}^n\|_{H^2}+\|\sigma^n\|_{W^{1,\infty}}+\|\sigma^n\|_{H^2}\le K. \tag{3.15}$$

Under induction assumptions (3.14)-(3.15) and according to Lemma 3.1, the hyperbolic equation (2.8) has a unique solution $\sigma^{n+1}\in H^2(\Omega)\cap L^\infty(\Omega)$ which satisfies

$$\sqrt{m}\le\min_{x\in\overline{\Omega}}\sigma^n(x)\le\sigma^{n+1}(x)\le\max_{x\in\overline{\Omega}}\sigma^n(x)\le\sqrt{M},\quad \forall x\in\Omega,\quad 0\le n\le N-1. \tag{3.16}$$

Thus, (3.2) is valid for $k=n+1$. Furthermore, (3.16) implies that

$$m\le\rho^{n+1}(x)\le M,\quad \forall x\in\Omega,\quad 0\le n\le N-1. \tag{3.17}$$

To close the mathematical induction, we need to prove that (3.3) is valid for $k=n+1$.

Step I: L^2 estimates of $\mathbf{e}_{\mathbf{u}}^{n+1}$ and $\mathbf{e}_{\mathbf{b}}^{n+1}$. Testing (3.13a) by $2\tau e_\sigma^{n+1}$, we have

$$\begin{aligned}
&\|e_\sigma^{n+1}\|_{L^2}^2-\|e_\sigma^n\|_{L^2}^2+\|e_\sigma^{n+1}-e_\sigma^n\|_{L^2}^2\\
=&2\tau(R_\sigma^{n+1},e_\sigma^{n+1})-2\tau(\nabla\sigma(t_{n+1})\cdot\mathbf{e}_{\mathbf{u}}^n,e_\sigma^{n+1})\\
\le&\frac{\tau}{2}\|e_\sigma^{n+1}\|_{L^2}^2+C\tau\|R_\sigma^{n+1}\|_{L^2}^2+C\tau\|\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2,
\end{aligned}$$

where we have noted (3.14) and

$$\begin{aligned}
2(\nabla\cdot(e_\sigma^{n+1}\mathbf{u}^n),e_\sigma^{n+1})=&2\int_\Omega\nabla e_\sigma^{n+1}\cdot\mathbf{u}^n e_\sigma^{n+1}dx=\int_\Omega\nabla|e_\sigma^{n+1}|^2\cdot\mathbf{u}^n dx\\
=&-\int_\Omega|e_\sigma^{n+1}|^2\nabla\cdot\mathbf{u}^n dx=0.
\end{aligned}$$

Summing up the above inequality and using the discrete Gronwall inequality in Lemma 3.3, we get

$$\|e_\sigma^{n+1}\|_{L^2}^2+\sum_{i=0}^n\|e_\sigma^{i+1}-e_\sigma^i\|_{L^2}^2\le C\tau^2+C\tau\sum_{i=0}^n\|\sigma^i\mathbf{e}_{\mathbf{u}}^i\|_{L^2}^2,\quad \forall 0\le n\le N-1. \tag{3.18}$$

Testing (3.13b) and (3.13c) by $2\tau\mathbf{e}_u^{n+1}$ and $2\tau S\mathbf{e}_b^{n+1}$, respectively, and adding the resulting equations, we obtain

$$\begin{aligned}
&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2-\|\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}-\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\frac{2\tau}{Re}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2\\
&+S\|\mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2-S\|\mathbf{e}_{\mathbf{b}}^n\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{n+1}-\mathbf{e}_{\mathbf{b}}^n\|_{L^2}^2+\frac{2\tau S}{Rm}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2\\
=&2\tau S(I_5^{n+1},\mathbf{e}_{\mathbf{b}}^{n+1})-2\tau\sum_{i=1}^4(I_i^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})+2\tau(\mathbf{R}_{\mathbf{u}}^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})+2\tau S(\mathbf{R}_{\mathbf{b}}^{n+1},\mathbf{e}_{\mathbf{b}}^{n+1}).
\end{aligned} \tag{3.19}$$

We estimate the right-hand side of (3.19) term by term by the Hölder inequality and the Young inequality. Firstly, from the integration by parts, one has

$$\left(\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{e}_{\mathbf{u}}^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1}\right)+\frac{1}{2}\left(\mathbf{e}_{\mathbf{u}}^{n+1}\nabla\cdot(\rho^{n+1}\mathbf{u}^n),\mathbf{e}_{\mathbf{u}}^{n+1}\right)=0.$$

Then using the regularity assumption (3.1) and induction assumptions (3.16)-(3.17), we can get

$$\begin{aligned}&-2\tau(I_1^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})-2\tau(I_3^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})\\=&-2\tau(e_\rho^{n+1}(\mathbf{u}(t_n)\cdot\nabla)\mathbf{u}(t_{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})-2\tau(\rho^{n+1}(\mathbf{e}_{\mathbf{u}}^{n}\cdot\nabla)\mathbf{u}(t_{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})\\&+\tau\left(e_\rho^{n+1}\mathbf{u}(t_n),\nabla(\mathbf{u}(t_{n+1})\cdot\mathbf{e}_{\mathbf{u}}^{n+1})\right)-\tau\left(\mathbf{u}(t_{n+1})\nabla\cdot(\rho^{n+1}\mathbf{e}_{\mathbf{u}}^{n}),\mathbf{e}_{\mathbf{u}}^{n+1}\right)\\\le& C\tau\left(\|e_\sigma^{n+1}\|_{L^2}^2+\|\sigma^n\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2\right)+\frac{\tau}{4Re}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2,\end{aligned}\tag{3.20}$$

where we use

$$\|e_\rho^{n+1}\|_{L^2}\le\|\sqrt{\rho(t_{n+1})}+\sigma^{n+1}\|_{L^\infty}\|e_\sigma^{n+1}\|_{L^2}\le C\|e_\sigma^{n+1}\|_{L^2}.\tag{3.21}$$

The term I_2^{n+1} can be bounded by

$$\begin{aligned}-2\tau(I_2^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})\le& C\tau(\|e_\sigma^{n+1}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}+\|e_\sigma^{n+1}\|_{L^2}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}+\|\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2})\\&+C\tau(\|R_\sigma^{n+1}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}+\|\nabla\sigma^{n+1}\|_{L^3}\|e_\sigma^{n+1}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^6})\\\le&\frac{\tau}{4Re}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+C\tau(\|e_\sigma^{n+1}\|_{L^2}^2+\|R_\sigma^{n+1}\|_{L^2}^2+\|\sigma^n\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2).\end{aligned}\tag{3.22}$$

Finally, from the vector formula (2.14), we have

$$\begin{aligned}&-2\tau(I_4^{n+1},\mathbf{e}_{\mathbf{u}}^{n+1})+2\tau S(I_5^{n+1},\mathbf{e}_{\mathbf{b}}^{n+1})\\=&2\tau S(\mathbf{u}(t_{n+1})\times\mathbf{e}_{\mathbf{b}}^{n},\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1})-2\tau S(\mathbf{e}_{\mathbf{b}}^{n}\times\mathbf{curl}\ \mathbf{b}(t_{n+1}),\mathbf{e}_{\mathbf{u}}^{n+1})\\\le& C\tau\|\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}\|\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}+C\tau\|\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}\\\le& C\tau\|\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}^2+\frac{\tau}{4Re}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\frac{\tau S}{2Rm}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2.\end{aligned}\tag{3.23}$$

Substituting (3.20), (3.22) and (3.23) into (3.19) yields

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}-\sigma^n\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2-\|\sigma^n\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2+\frac{\tau}{Re}\|\nabla\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2\\&+S\|\mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{n+1}-\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}^2-S\|\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}^2+\frac{\tau S}{Rm}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2\\\le& C\tau\left(\|e_\sigma^{n+1}\|_{L^2}^2+\|\sigma^n\mathbf{e}_{\mathbf{u}}^{n}\|_{L^2}^2+\|\mathbf{e}_{\mathbf{b}}^{n}\|_{L^2}^2+\|\mathbf{R}_{\mathbf{u}}^{n+1}\|_{L^2}^2+\|\mathbf{R}_{\mathbf{b}}^{n+1}\|_{L^2}^2+\|R_\sigma^{n+1}\|_{L^2}^2\right).\end{aligned}$$

Summing up the above inequality and using (3.18) yields

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2+\sum_{i=0}^{n}\Big(\|\sigma^{i+1}\mathbf{e}_{\mathbf{u}}^{i+1}-\sigma^i\mathbf{e}_{\mathbf{u}}^i\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{i+1}-\mathbf{e}_{\mathbf{b}}^i\|_{L^2}^2\Big)\\&\quad+\frac{\tau}{Re}\sum_{i=0}^{n}\|\nabla\mathbf{e}_{\mathbf{u}}^{i+1}\|_{L^2}^2+\frac{S\tau}{Rm}\sum_{i=0}^{n}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{i+1}\|_{L^2}^2\\&\le C\tau^2+C\tau\sum_{i=0}^{n}(\|\sigma^i\mathbf{e}_{\mathbf{u}}^i\|_{L^2}^2+\|\mathbf{e}_{\mathbf{b}}^i\|_{L^2}^2),\quad \forall\, 0\le n\le N-1.\end{aligned}$$

By the Gronwall inequality in Lemma 3.3, we get

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{n+1}\|_{L^2}^2+\sum_{i=0}^{n}\Big(\|\sigma^{i+1}\mathbf{e}_{\mathbf{u}}^{i+1}-\sigma^i\mathbf{e}_{\mathbf{u}}^i\|_{L^2}^2+S\|\mathbf{e}_{\mathbf{b}}^{i+1}-\mathbf{e}_{\mathbf{b}}^i\|_{L^2}^2\Big)\\&\quad+\frac{\tau}{Re}\sum_{i=0}^{n}\|\nabla\mathbf{e}_{\mathbf{u}}^{i+1}\|_{L^2}^2+\frac{S\tau}{Rm}\sum_{i=0}^{n}\|\mathbf{curl}\ \mathbf{e}_{\mathbf{b}}^{i+1}\|_{L^2}^2\le C\tau^2,\quad \forall 0\le n\le N-1,\end{aligned}\tag{3.24}$$

which with (3.18) yields

$$\|e_\sigma^{n+1}\|_{L^2}^2+\sum_{i=0}^{n}\|e_\sigma^{i+1}-e_\sigma^i\|_{L^2}^2\le C\tau^2,\quad \forall 0\le n\le N-1.\tag{3.25}$$

Step II: H^2 **regularities of** $\mathbf{u}^{n+1}$ **and** $\mathbf{b}^{n+1}$**.** The inequality (3.24) implies the uniform boundness of $\mathbf{u}^{n+1}$ and $\mathbf{b}^{n+1}$ in H^1-norm, i.e., there exists some $C>0$ such that

$$\|\nabla\mathbf{u}^{n+1}\|_{L^2}+\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}\le C.\tag{3.26}$$

On the other hand, from

$$\|\sigma^{n+1}-\sigma^n\|_{L^2}\le\|\sigma(t_{n+1})-\sigma(t_n)\|_{L^2}+\|e_\sigma^{n+1}-e_\sigma^n\|_{L^2}\le C\tau\tag{3.27}$$

and

$$\begin{aligned}&\|\sigma^{n+1}\mathbf{u}^{n+1}-\sigma^n\mathbf{u}^n\|_{L^2}\\&\le\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}-\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}+\|(\sigma^{n+1}-\sigma^n)\mathbf{u}(t_{n+1})\|_{L^2}+\|\sigma^n(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\|_{L^2},\end{aligned}$$

we have

$$\begin{aligned}&\sum_{n=0}^{m}\|\sigma^{n+1}\mathbf{u}^{n+1}-\sigma^n\mathbf{u}^n\|_{L^2}^2\\&\le C\sum_{n=0}^{m}\|\sigma^{n+1}\mathbf{e}_{\mathbf{u}}^{n+1}-\sigma^n\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+C\sum_{n=0}^{m}\|\sigma^{n+1}-\sigma^n\|_{L^2}^2\\&\quad+C\sum_{n=0}^{m}\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_{L^2}^2\le C\tau.\end{aligned}\tag{3.28}$$

In order to derive the H^2 regularities of the numerical solution, we rewrite (2.9) and (2.10) as the following Stokes type and Maxwell type problems:

$$\begin{aligned}
&-\frac{1}{Re}\Delta\mathbf{u}^{n+1}+\nabla p^{n+1}\\
=&-\sigma^{n+1}D_\tau(\sigma^{n+1}\mathbf{u}^{n+1})-\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\\
&-\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)-S(\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1})+\mathbf{f}^{n+1},\quad \nabla\cdot\mathbf{u}^{n+1}=0,
\end{aligned}\tag{3.29}$$

and

$$\frac{1}{Rm}\mathbf{curl}\ (\mathbf{curl}\ \mathbf{b}^{n+1})=\mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n)-D_\tau\mathbf{b}^{n+1},\quad \nabla\cdot\mathbf{b}^{n+1}=0.\tag{3.30}$$

Next, we estimate the right-hand side of (3.29) term by term. From (3.16) and (3.28), we have

$$\tau\sum_{n=0}^{N-1}\|\sigma^{n+1}D_\tau(\sigma^{n+1}\mathbf{u}^{n+1})\|_{L^2}^2\le C.\tag{3.31}$$

In terms of the Sobolev embedding theorem, (3.17) and (3.26), we can get

$$\tau\sum_{n=0}^{N-1}\|\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\|_{L^{3/2}}^2\le C\tau\sum_{n=0}^{N-1}\|\rho^{n+1}\|_{L^\infty}^2\|\nabla\mathbf{u}^n\|_{L^2}^2\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2\le C,\tag{3.32a}$$

$$\tau\sum_{n=0}^{N-1}\|S\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^{3/2}}^2\le C\tau\sum_{n=0}^{N-1}\|\mathbf{curl}\ \mathbf{b}^n\|_{L^2}^2\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}^2\le C.\tag{3.32b}$$

According to (2.8), we have

$$-\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)=-\sigma^{n+1}\mathbf{u}^{n+1}\nabla\cdot(\sigma^{n+1}\mathbf{u}^n)=\sigma^{n+1}D_\tau\sigma^{n+1}\mathbf{u}^{n+1}.$$

Then

$$\begin{aligned}
&\left\|\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)\right\|_{L^{3/2}}\\
=&\|\sigma^{n+1}\mathbf{u}^{n+1}D_\tau\sigma^{n+1}\|_{L^{3/2}}\\
\le& C\tau^{-1}\|\mathbf{u}^{n+1}\|_{L^6}\|\sigma^{n+1}-\sigma^n\|_{L^2}.
\end{aligned}\tag{3.33}$$

From (3.27), one has

$$\tau\sum_{n=0}^{N-1}\left\|\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)\right\|_{L^{3/2}}^2\le C.\tag{3.34}$$

Applying the regularity of the Stokes problem to (3.29), and using (3.31)-(3.32b) and (3.34), we have

$$\tau\sum_{n=0}^{N-1}\|\mathbf{u}^{n+1}\|^2_{W^{2,3/2}}\le C.$$

According to the Sobolev imbedding $\mathbf{W}^{2,3/2}(\Omega)\subset\mathbf{W}^{1,3}(\Omega)$, we get

$$\tau\sum_{n=0}^{N-1}\|\mathbf{u}^{n+1}\|^2_{W^{1,3}}\le C. \tag{3.35}$$

By a similar method, one has

$$\|\mathbf{b}^{n+1}-\mathbf{b}^n\|_{L^2}\le\|\mathbf{b}(t_{n+1})-\mathbf{b}(t_n)\|_{L^2}+\|\mathbf{e}_{\mathbf{b}}^{n+1}-\mathbf{e}_{\mathbf{b}}^n\|_{L^2},$$

which with (3.24) leads to

$$\tau\sum_{n=0}^{N-1}\|D_\tau\mathbf{b}^{n+1}\|^2_{L^2}\le C. \tag{3.36}$$

Furthermore, from (3.26), one has

$$\begin{aligned}\|\mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n)\|_{L^{3/2}}\le& C\|\nabla\mathbf{u}^{n+1}\|_{L^2}\|\mathbf{b}^n\|_{L^6}+C\|\mathbf{u}^{n+1}\|_{L^6}\|\nabla\mathbf{b}^n\|_{L^2}\\ \le& C\|\nabla\mathbf{u}^{n+1}\|_{L^2}\|\mathbf{b}^n\|_{L^6}+C\|\mathbf{u}^{n+1}\|_{L^6}\|\mathbf{curl}\ \mathbf{b}^n\|_{L^2}\le C,\end{aligned}$$

where we noted

$$\|\nabla\mathbf{b}^n\|_{L^2}\le\|\mathbf{b}^n\|_{H^1}\le C\|\mathbf{curl}\ \mathbf{b}^n\|_{L^2}.$$

Thus,

$$\tau\sum_{n=0}^{N-1}\|\mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n)\|^2_{L^{3/2}}\le C. \tag{3.37}$$

Applying the regularity theory of the Maxwell equation to (3.30) and using (3.36)-(3.37) yield

$$\tau\sum_{n=0}^{N-1}\|\mathbf{b}^{n+1}\|^2_{W^{2,3/2}}\le C.$$

By the Sobolev imbedding $\mathbf{W}^{2,3/2}(\Omega)\subset\mathbf{W}^{1,3}(\Omega)$, again, we get

$$\tau\sum_{n=0}^{N-1}\|\mathbf{b}^{n+1}\|^2_{W^{1,3}}\le C. \tag{3.38}$$

Based on (3.35) and (3.38), we can obtain the H^2 regularities of numerical solution $(\mathbf{u}^{n+1},\mathbf{b}^{n+1})$. Firstly, one has

$$
\begin{aligned}
&\|\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\|_{L^2}\le C\|\mathbf{u}^n\|_{L^6}\|\nabla\mathbf{u}^{n+1}\|_{L^3}\le C\|\mathbf{u}^{n+1}\|_{W^{1,3}},\\
&\|\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}\le C\|\mathbf{b}^n\|_{L^6}\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^3}\le C\|\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^3},\\
&\|\mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n)\|_{L^2}\le C\|\nabla\mathbf{u}^{n+1}\|_{L^3}\|\mathbf{b}^n\|_{L^6}+C\|\mathbf{u}^{n+1}\|_{L^6}\|\mathbf{curl}\ \mathbf{b}^n\|_{L^3}\\
&\qquad\qquad\le C(\|\nabla\mathbf{u}^{n+1}\|_{L^3}+\|\mathbf{curl}\ \mathbf{b}^n\|_{L^3}).
\end{aligned}
$$

Then

$$
\tau\sum_{n=0}^{N-1}\Big(\|\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{b}^n\times\mathbf{curl}\ \mathbf{b}^{n+1}\|_{L^2}^2+\|\mathbf{curl}\ (\mathbf{u}^{n+1}\times\mathbf{b}^n)\|_{L^2}^2\Big)\le C. \tag{3.39}
$$

By the Agmon inequality, (3.16) and (3.27), we have

$$
\begin{aligned}
\left\|\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)\right\|_{L^2}^2&=\|\sigma^{n+1}\mathbf{u}^{n+1}D_\tau\sigma^{n+1}\|_{L^2}^2\le C\|\mathbf{u}^{n+1}\|_{L^\infty}^2\|D_\tau\sigma^{n+1}\|_{L^2}^2\\
&\le C\|\nabla\mathbf{u}^{n+1}\|_{L^2}\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}\le\frac{1}{2}\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}^2+C\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2.
\end{aligned}\tag{3.40}
$$

The regularity theories of the Stokes and Maxwell problems with (3.31), (3.36), (3.39) and (3.40) yields

$$
\begin{aligned}
&\tau\sum_{n=0}^{N-1}\Big(\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}^2+\|p^{n+1}\|_{H^1}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2\Big)\\
\le&\tau\sum_{n=0}^{N-1}\left\|\frac{\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)\right\|_{L^2}^2+C\\
\le&\frac{\tau}{2}\sum_{n=0}^{N-1}\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}^2+C.
\end{aligned}
$$

Thus, we get the regularities of numerical solution:

$$
\tau\sum_{n=0}^{N-1}\Big(\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}^2+\|p^{n+1}\|_{H^1}^2+\|\mathbf{b}^{n+1}\|_{H^2}^2\Big)\le C. \tag{3.41}
$$

Step III: H^1 **estimates of** $\mathbf{e}_{\mathbf{u}}^{n+1}$ **and** $\mathbf{e}_{\mathbf{b}}^{n+1}$**.** For $j=1,2,3$, differentiating (3.13a) with respect to x_j gives

$$
\begin{aligned}
&\frac{D_je_\sigma^{n+1}-D_je_\sigma^n}{\tau}+\nabla(D_j\sigma(t_{n+1}))\cdot\mathbf{e}_{\mathbf{u}}^n+\nabla\sigma(t_{n+1})\cdot D_j\mathbf{e}_{\mathbf{u}}^n\\
&\quad+\nabla(D_je_\sigma^{n+1})\cdot\mathbf{u}^n+\nabla e_\sigma^{n+1}\cdot D_j\mathbf{u}^n=D_jR_\sigma^{n+1},
\end{aligned}\tag{3.42}
$$

where the differential operator $D_j=\partial/\partial x_j$. Testing the above equation by $2\tau D_j e_\sigma^{n+1}$ yields

$$\begin{aligned}&\|D_j e_\sigma^{n+1}\|_{L^2}^2+\|D_j e_\sigma^{n+1}-D_j e_\sigma^n\|_{L^2}^2-\|D_j e_\sigma^n\|_{L^2}^2\\ \le& C\tau(\|\mathbf{e}_\mathbf{u}^n\|_{L^4}^2+\|D_j\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|D_j R_\sigma^{n+1}\|_{L^2}^2)+C\tau\|D_j e_\sigma^{n+1}\|_{L^2}^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla(e_\sigma^{n+1}-e_\sigma^n)\|_{L^2}^2-\|\nabla e_\sigma^n\|_{L^2}^2\\ \le& C\tau(\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla R_\sigma^{n+1}\|_{L^2}^2)+C\tau\|\nabla e_\sigma^{n+1}\|_{L^2}^2.\end{aligned}$$

Summing up the above inequality and using the discrete Gronwall inequality, we get

$$\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\sum_{i=0}^n\|\nabla(e_\sigma^{i+1}-e_\sigma^i)\|_{L^2}^2\le C\tau\sum_{i=0}^n(\|\nabla\mathbf{e}_\mathbf{u}^i\|_{L^2}^2+\|\nabla R_\sigma^{i+1}\|_{L^2}^2)\le C\tau^2. \tag{3.43}$$

To derive the H^1 error estimate of the velocity field, we rewrite the error equation (3.13b) as follows:

$$\rho^{n+1}D_\tau\mathbf{e}_\mathbf{u}^{n+1}-\frac{1}{Re}\Delta\mathbf{e}_\mathbf{u}^{n+1}+\nabla e_p^{n+1}+\sum_{i=1}^4 I_i^{n+1}+I_6^{n+1}=\mathbf{R}_\mathbf{u}^{n+1}, \tag{3.44}$$

where $I_1^{n+1},\cdots,I_4^{n+1}$ are defined as before, and I_6^{n+1} is given by

$$I_6^{n+1}=\sigma^{n+1}D_\tau(\sigma^{n+1}\mathbf{e}_\mathbf{u}^{n+1})-\rho^{n+1}D_\tau\mathbf{e}_\mathbf{u}^{n+1}=-\sigma^{n+1}\nabla\cdot(\sigma^{n+1}\mathbf{u}^n)\mathbf{e}_\mathbf{u}^n.$$

Multiplying (3.44) by $2\tau D_\tau\mathbf{e}_\mathbf{u}^{n+1}$ and integrating over Ω yields

$$\begin{aligned}&2\tau\|\sigma^{n+1}D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\frac{1}{Re}\Big(\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\nabla\mathbf{e}_\mathbf{u}^{n+1}-\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2-\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2\Big)\\ =&2\tau(\mathbf{R}_\mathbf{u}^{n+1},D_\tau\mathbf{e}_\mathbf{u}^{n+1})-2\tau\sum_{i=1}^4(I_i^{n+1},D_\tau\mathbf{e}_\mathbf{u}^{n+1})-2\tau(I_6^{n+1},D_\tau\mathbf{e}_\mathbf{u}^{n+1}),\end{aligned}$$

which with (3.16) leads to

$$\begin{aligned}&2m\tau\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\frac{1}{Re}\Big(\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\nabla\mathbf{e}_\mathbf{u}^{n+1}-\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2-\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2\Big)\\ \le&\frac{m\tau}{2}\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+2\tau|(I_3^{n+1},D_\tau\mathbf{e}_\mathbf{u}^{n+1})|\\ &+C\tau(\|I_1^{n+1}\|_{L^2}^2+\|I_2^{n+1}\|_{L^2}^2+\|I_4^{n+1}\|_{L^2}^2+\|I_6^{n+1}\|_{L^2}^2+\|\mathbf{R}_\mathbf{u}^{n+1}\|^2).\end{aligned} \tag{3.45}$$

We can estimate $\|I_1^{n+1}\|_{L^2}^2$, $\|I_2^{n+1}\|_{L^2}^2$, $\|I_4^{n+1}\|_{L^2}^2$ and $\|I_6^{n+1}\|_{L^2}^2$ as follows:

$$\begin{aligned}&\|I_1^{n+1}\|_{L^2}^2\le C(\|e_\sigma^{n+1}\|_{L^2}^2+\|\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\mathbf{u}^n\|_{H^2}^2\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2),\\ &\|I_2^{n+1}\|_{L^2}^2\le C(\|e_\sigma^{n+1}\|_{L^2}^2+\|\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\mathbf{u}^n\|_{H^2}^2\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|R_\sigma^{n+1}\|_{L^2}^2),\\ &\|I_4^{n+1}\|_{L^2}^2\le C(\|\mathbf{e}_\mathbf{b}^n\|_{L^2}^2+\|\mathbf{curl}\ \mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}^2),\\ &\|I_6^{n+1}\|_{L^2}^2\le C\|\mathbf{u}^n\|_{H^2}^2\|\nabla\sigma^{n+1}\|_{L^3}^2\|\mathbf{e}_\mathbf{u}^n\|_{L^6}^2\le C\|\mathbf{u}^n\|_{H^2}^2\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2,\end{aligned}$$

where (3.1), (3.17) and (3.21) are used.

For the term I_3^{n+1} in (3.45), we rewrite I_3 as follows:

$$\begin{aligned}I_3^{n+1}=&\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(e_\rho^{n+1}\mathbf{u}(t_n))+\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(\rho^{n+1}\mathbf{e}_\mathbf{u}^n)+\frac{\mathbf{e}_\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho^{n+1}\mathbf{u}^n)\\ =&\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(e_\rho^{n+1}\mathbf{u}(t_n))+\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(\rho(t_{n+1})\mathbf{e}_\mathbf{u}^n)-\frac{\mathbf{u}(t_{n+1})}{2}\nabla\cdot(e_\rho^{n+1}\mathbf{e}_\mathbf{u}^n)\\ &+\frac{\mathbf{e}_\mathbf{u}^{n+1}}{2}\nabla\cdot(\rho(t_{n+1})\mathbf{u}^n)-\frac{\mathbf{e}_\mathbf{u}^{n+1}}{2}\nabla\cdot(e_\rho^{n+1}\mathbf{u}^n).\end{aligned}\tag{3.46}$$

Then there holds

$$\begin{aligned}&2\tau|(I_3^{n+1},D_\tau\mathbf{e}_\mathbf{u}^{n+1})|\\ \le& C\tau\|\mathbf{u}(t_{n+1})\nabla e_\rho^{n+1}\cdot\mathbf{u}(t_n)\|_{L^2}\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}+C\tau\|\mathbf{u}(t_{n+1})\nabla\rho(t_{n+1})\cdot\mathbf{e}_\mathbf{u}^n\|_{L^2}\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}\\ &+C\tau\|\mathbf{e}_\mathbf{u}^{n+1}\nabla\rho(t_{n+1})\cdot\mathbf{u}^n\|_{L^2}\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}+C|(\mathbf{u}(t_{n+1})\nabla\cdot(e_\rho^{n+1}\mathbf{e}_\mathbf{u}^n),\mathbf{e}_\mathbf{u}^{n+1}-\mathbf{e}_\mathbf{u}^n)|\\ &+C|(\mathbf{e}_\mathbf{u}^{n+1}\nabla\cdot(e_\rho^{n+1}\mathbf{u}^n),\mathbf{e}_\mathbf{u}^{n+1}-\mathbf{e}_\mathbf{u}^n)|\\ \le&\frac{m\tau}{2}\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+C\tau(\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2)\\ &+\frac{1}{2Re}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}-\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+C(\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2)\|e_\sigma^{n+1}\|_{H^1}^2,\end{aligned}$$

where we use

$$\|\nabla e_\rho^{n+1}\|_{L^2}=\|\nabla(e_\sigma^{n+1}(2\sigma(t_{n+1})-e_\sigma^{n+1}))\|_{L^2}\le C\|e_\sigma^{n+1}\|_{H^1}.\tag{3.47}$$

Substituting the above inequalities into (3.45) and using (3.43), we get

$$\begin{aligned}&m\tau\sum_{i=0}^n\|D_\tau\mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^2+\frac{1}{Re}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\frac{1}{2Re}\sum_{i=0}^n\|\nabla\mathbf{e}_\mathbf{u}^{i+1}-\nabla\mathbf{e}_\mathbf{u}^i\|_{L^2}^2\\ \le& C\tau^2+C\tau\sum_{i=0}^n\|\mathbf{u}^i\|_{H^2}^2\|\nabla\mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^2+C\tau\sum_{i=0}^n\|\nabla\mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^2,\end{aligned}$$

which with the discrete Gronwall inequality gives

$$m\tau\sum_{i=0}^n\|D_\tau\mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^2+\frac{1}{Re}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\frac{1}{2Re}\sum_{i=0}^n\|\nabla\mathbf{e}_\mathbf{u}^{i+1}-\nabla\mathbf{e}_\mathbf{u}^i\|_{L^2}^2.\le C\tau^2.\tag{3.48}$$

Applying the classical regularities theory of the Stokes problem to (3.44) and using (3.24), (3.25), (3.43) and (3.48), we get

$$\begin{aligned}&\tau\sum_{n=0}^m(\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\nabla e_p^{n+1}\|_{L^2}^2)\\ \le& C\tau\sum_{n=0}^m\left(\|D_\tau\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\sum_{i=1}^4\|I_i^{n+1}\|_{L^2}^2+\|I_6^{n+1}\|_{L^2}^2+\|\mathbf{R}_\mathbf{u}^{n+1}\|_{L^2}^2\right)\le C\tau^2\end{aligned}\tag{3.49}$$

by noting

$$\begin{aligned}\tau\|I_3^{n+1}\|_{L^2}^2 \leq & C\tau(\|\nabla e_\rho^{n+1}\|_{L^2}^2+\|\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla e_\rho^{n+1}\|_{L^2}^2\|\mathbf{e}_\mathbf{u}^n\|_{L^\infty}^2)\\ &+C\tau(\|\mathbf{u}^n\|_{H^2}^2\|\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{u}^n\|_{H^2}^2\|\nabla e_\rho^{n+1}\|_{L^2}^2\|\mathbf{e}_\mathbf{u}^{n+1}\|_{L^\infty}^2)\\ \leq & C\tau(\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^2}^2\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2})\\ &+C\tau(\|\mathbf{u}^n\|_{H^2}^2\|\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{u}^n\|_{H^2}^2\|\nabla e_\sigma^{n+1}\|_{L^2}^2\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2})\\ \leq & C\tau(\|\nabla e_\sigma^{n+1}\|_{L^2}^2+\|\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^2}^4\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2)\\ &+C\tau(\|\mathbf{u}^n\|_{H^2}^2\|\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{u}^n\|_{H^2}^4\|\nabla e_\sigma^{n+1}\|_{L^2}^4\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2)+\epsilon\tau\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2}^2\end{aligned}$$

and

$$\tau\sum_{n=0}^m\|I_3^{n+1}\|_{L^2}^2\leq\epsilon\tau\sum_{n=0}^m\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+C\tau^2$$

for some sufficiently small $\epsilon>0$.

By the similar method for the estimate of $\mathbf{e}_\mathbf{u}^{n+1}$, we make the error analysis for $\mathbf{e}_\mathbf{b}^{n+1}$. The Eq. (3.13c) can be written as

$$D_\tau\mathbf{e}_\mathbf{b}^{n+1}+\frac{1}{Rm}\mathbf{curl}\;(\mathbf{curl}\;\mathbf{e}_\mathbf{b}^{n+1})=I_5^{n+1}+\mathbf{R}_\mathbf{b}^{n+1},\quad \nabla\cdot\mathbf{e}_\mathbf{b}^{n+1}=0, \tag{3.50}$$

where the term I_5^{n+1} is rewritten as

$$\begin{aligned}I_5^{n+1}=&\mathbf{curl}\;(\mathbf{u}(t_{n+1})\times\mathbf{e}_\mathbf{b}^n)+\mathbf{curl}\;(\mathbf{e}_\mathbf{u}^{n+1}\times\mathbf{b}^n)\\ =&(\mathbf{e}_\mathbf{b}^n\cdot\nabla)\mathbf{u}(t_{n+1})-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{e}_\mathbf{b}^n+(\mathbf{b}^n\cdot\nabla)\mathbf{e}_\mathbf{u}^{n+1}-(\mathbf{e}_\mathbf{u}^{n+1}\cdot\nabla)\mathbf{b}^n.\end{aligned} \tag{3.51}$$

Multiplying (3.50) by $2\tau D_\tau\mathbf{e}_\mathbf{b}^{n+1}$ and integrating over Ω yields

$$\begin{aligned}&2\tau\|D_\tau\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}^2+\frac{1}{Rm}\Big(\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}^2-\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^n\|_{L^2}^2+\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^{n+1}-\mathbf{curl}\;\mathbf{e}_\mathbf{b}^n\|_{L^2}^2\Big)\\ \leq& C\tau\Big(\|\mathbf{e}_\mathbf{b}^n\|_{L^2}+\|\nabla\mathbf{e}_\mathbf{b}^n\|_{L^2}+\|\mathbf{b}^n\|_{H^2}\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}\Big)\|D_\tau\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}\\ &+C\tau\Big(\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}\|\nabla\mathbf{b}^n\|_{L^2}+\|\mathbf{R}_\mathbf{b}^{n+1}\|_{L^2}\Big)\|D_\tau\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}\\ \leq&\tau\|D_\tau\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}^2+C\tau\big(\|\mathbf{e}_\mathbf{b}^n\|_{L^2}^2+\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^n\|_{L^2}^2\big)\\ &+C\tau\Big(\|\mathbf{b}^n\|_{H^2}^2\|\nabla\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2+\|\mathbf{R}_\mathbf{b}^{n+1}\|_{L^2}^2\Big).\end{aligned}$$

From (3.24), (3.41) and (3.49), we have

$$\begin{aligned}&\tau\sum_{i=0}^n\|D_\tau\mathbf{e}_\mathbf{b}^{i+1}\|_{L^2}^2+\frac{1}{Rm}\left(\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^{n+1}\|_{L^2}^2+\sum_{i=0}^n\|\mathbf{curl}\;\mathbf{e}_\mathbf{b}^{i+1}-\mathbf{curl}\;\mathbf{e}_\mathbf{b}^i\|_{L^2}^2\right)\\ \leq& C\tau^2,\quad\forall 0\leq n\leq N-1.\end{aligned} \tag{3.52}$$

In addition, $\|I_5^{n+1}\|_{L^2}^2$ can be estimated by

$$\|I_5^{n+1}\|_{L^2}^2 \le C\left(\|\mathbf{e}_\mathbf{b}^n\|_{L^2}^2 + \|\mathbf{curl}\ \mathbf{e}_\mathbf{b}^n\|_{L^2}^2 + \|\mathbf{b}^n\|_{H^2}^2\|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2 + \|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2\right). \tag{3.53}$$

Then the classic regularity theory of the Maxwell equation implies that

$$\tau\sum_{i=0}^n \|\mathbf{e}_\mathbf{b}^{i+1}\|_{H^2}^2 \le C\tau\sum_{i=0}^n \left(\|D_\tau \mathbf{e}_\mathbf{b}^{i+1}\|_{L^2}^2 + \|I_5^{i+1}\|_{L^2}^2 + \|\mathbf{R}_\mathbf{b}^{i+1}\|_{L^2}^2\right) \le C\tau^2. \tag{3.54}$$

Step IV: $W^{1,4}$ **estimate of** e_σ^{n+1}**.** Thanks to (4.35) in [9], one has

$$\|e_\sigma^{n+1}\|_{L^\infty} \le C\tau\sum_{i=0}^n (\|R_\sigma^{i+1}\|_{L^\infty} + \|\mathbf{e}_\mathbf{u}^i\|_{L^\infty}) \le C\tau. \tag{3.55}$$

Testing (3.42) by $\tau|D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}$, we have

$$\begin{aligned}
&\|D_j e_\sigma^{n+1}\|_{L^4}^4 - \left(D_j e_\sigma^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right) + \tau\left(\nabla(D_j\sigma(t_{n+1}))\cdot \mathbf{e}_\mathbf{u}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right)\\
&\quad + \tau\left(\nabla\sigma(t_{n+1})\cdot D_j\mathbf{e}_\mathbf{u}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right) + \tau\left(\nabla(D_j e_\sigma^{n+1})\cdot \mathbf{u}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right)\\
&\quad + \tau\left(\nabla e_\sigma^{n+1}\cdot D_j\mathbf{u}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right) = \tau\left(D_j R_\sigma^{n+1}, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}\right),
\end{aligned}$$

where we use

$$(\nabla(D_j e_\sigma^{n+1})\cdot \mathbf{u}^n, |D_j e_\sigma^{n+1}|^2 D_j e_\sigma^{n+1}) = -\frac{1}{4}(|D_j e_\sigma^{n+1}|^4, \nabla\cdot\mathbf{u}^n) = 0.$$

Furthermore, from the Hölder inequality, one has

$$\begin{aligned}
&\|D_j e_\sigma^{n+1}\|_{L^4}^4 - \|D_j e_\sigma^n\|_{L^4}\|D_j e_\sigma^{n+1}\|_{L^4}^3\\
\le& C\tau\left(\|\nabla(D_j\sigma(t_{n+1}))\cdot \mathbf{e}_\mathbf{u}^n\|_{L^4} + \|\nabla\sigma(t_{n+1})\cdot D_j\mathbf{e}_\mathbf{u}^n\|_{L^4}\right)\|D_j e_\sigma^{n+1}\|_{L^4}^3\\
&+ C\tau\left(\|\nabla e_\sigma^{n+1}\cdot D_j\mathbf{u}^n\|_{L^4} + \|D_j R_\sigma^{n+1}\|_{L^4}\right)\|D_j e_\sigma^{n+1}\|_{L^4}^3,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|D_j e_\sigma^{n+1}\|_{L^4} \le& C\tau\sum_{i=0}^n (\|\nabla(D_j\sigma(t_{i+1}))\cdot \mathbf{e}_\mathbf{u}^i\|_{L^4} + \|\nabla\sigma(t_{i+1})\cdot D_j\mathbf{e}_\mathbf{u}^i\|_{L^4})\\
&+ C\tau\sum_{i=0}^n (\|\nabla e_\sigma^{i+1}\cdot D_j\mathbf{u}^i\|_{L^4} + \|D_j R_\sigma^{i+1}\|_{L^4})\\
\le& C\tau\sum_{i=0}^n \left(\|\mathbf{A}\mathbf{e}_\mathbf{u}^i\|_{L^2} + \|D_j\mathbf{e}_\mathbf{u}^i\|_{H^1} + \|\nabla e_\sigma^{i+1}\|_{L^4} + \|D_j R_\sigma^{i+1}\|_{L^4}\right).
\end{aligned} \tag{3.56}$$

Using the discrete Gronwall inequality, (3.24) and (3.49), we can derive that

$$\|\nabla e_\sigma^{n+1}\|_{L^4} \le C\tau \sum_{i=0}^{n} (\|\mathbf{A}\mathbf{e}_\mathbf{u}^i\|_{L^2} + \|\nabla \mathbf{e}_\mathbf{u}^i\|_{L^2} + \|\nabla R_\sigma^{i+1}\|_{L^4}) \le C\tau \tag{3.57}$$

for all $0 \le n \le N-1$.

Step V: $W^{1,\infty}$ **estimate of** $\mathbf{e}_\mathbf{u}^{n+1}$**.** In terms of the Helmholtz-Weyl decomposition (2.1), there has a unique decomposition

$$\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1} = \mathbb{P}(\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1}) + \nabla \phi^{n+1}, \tag{3.58}$$

where ϕ^{n+1} with $\int_\Omega \phi^{n+1} dx = 0$ is the solution to the following Neumann problem:

$$\Delta \phi^{n+1} = \nabla \cdot (\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1}) \quad \text{in } \Omega, \tag{3.59}$$

with the boundary condition:

$$\nabla \phi^{n+1} \cdot \mathbf{n} = (\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma. \tag{3.60}$$

By the regularity of solution to the Neumann problem, the solution ϕ^{n+1} satisfies

$$\begin{aligned} \|\nabla \phi^{n+1}\|_{L^4} &\le C\|\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1}\|_{L^4} \\ &\le C\|D_\tau \mathbf{e}_\mathbf{u}^{n+1}\|_{L^4} \\ &\le C\|D_\tau \mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^{\frac{1}{4}} \|\nabla (D_\tau \mathbf{e}_\mathbf{u}^{n+1})\|_{L^2}^{\frac{3}{4}}. \end{aligned}$$

Then

$$\begin{aligned} \tau \sum_{i=0}^{n} \|\nabla \phi^{i+1}\|_{L^4}^2 &\le C\tau \sum_{i=0}^{n} \left(\|D_\tau \mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^{\frac{1}{2}} \|\nabla (D_\tau \mathbf{e}_\mathbf{u}^{i+1})\|_{L^2}^{\frac{3}{2}} \right) \\ &\le C\tau^{\frac{1}{2}} \sum_{i=0}^{n} \|D_\tau \mathbf{e}_\mathbf{u}^{i+1}\|_{L^2}^2 + C\tau^{\frac{7}{6}} \sum_{i=0}^{n} \|\nabla (D_\tau \mathbf{e}_\mathbf{u}^{i+1})\|_{L^2}^2 \le C\tau^{\frac{7}{6}}, \end{aligned} \tag{3.61}$$

where (3.48) is used.

Applying the projection operator $\mathbb{P}$ to (3.44), we get

$$\rho^{n+1} D_\tau \mathbf{e}_\mathbf{u}^{n+1} + \frac{1}{Re} \mathbf{A}\mathbf{e}_\mathbf{u}^{n+1} = \nabla \phi^{n+1} + \mathbb{P}\left(\mathbf{R}_\mathbf{u}^{n+1} - \sum_{i=1}^{4} I_i^{n+1} - I_6^{n+1} \right), \tag{3.62}$$

where I_i^{n+1}, $i = 1, \cdots, 4$, and I_6^{n+1} are defined as before. In order to get the estimate of $\|\mathbf{e}_\mathbf{u}^{n+1}\|_{W^{1,\infty}}$, by the Sobolev imbedding $\mathbf{W}^{2,4}(\Omega) \subset \mathbf{W}^{1,\infty}(\Omega)$, we need to derive the estimate $\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^4}$ by using Lemma 3.2. Firstly, we check the conditions in Lemma 3.2 as follows.

The first condition holds from (3.17). For the second condition, one has

$$\begin{aligned}\|\rho^{n+1}\|_{W^{1,4}} \le& \|e_\rho^{n+1}\|_{W^{1,4}} + \|\rho(t_{n+1})\|_{W^{1,4}}\\ \le& C\|e_\sigma^{n+1}\|_{W^{1,4}} + \|\rho(t_{n+1})\|_{W^{1,4}} \le C, \quad \forall 0 \le n \le N-1. \end{aligned}\tag{3.63}$$

For the last condition, by (3.1) and (3.55), it is easy to see that

$$\begin{aligned}\sum_{n=0}^{N-1} \|\rho^{n+1} - \rho^n\|_{L^\infty} \le& \sum_{n=0}^{N-1} \Big(\|e_\rho^{n+1} - e_\rho^n\|_{L^\infty} + \|\rho(t_{n+1}) - \rho(t_n)\|_{L^\infty} \Big)\\ \le& C\sum_{n=0}^{N-1} \|e_\sigma^{n+1} - e_\sigma^n\|_{L^\infty} + C \le C. \end{aligned}\tag{3.64}$$

Thus, all conditions in Lemma 3.2 hold. Then from Lemma 3.2, we have

$$\begin{aligned}&\tau \sum_{n=0}^{N-1} (\|D_\tau \mathbf{e}_\mathbf{u}^{n+1}\|_{L^4}^2 + \|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^4}^2)\\ \le& C\tau \sum_{n=0}^{N-1} \left(\|\nabla\phi^{n+1}\|_{L^4}^2 + \sum_{j=1}^{4} \|I_j^{n+1}\|_{L^4}^2 + \|I_6^{n+1}\|_{L^4}^2 + \|\mathbf{R}_\mathbf{u}^{n+1}\|_{L^4}^2 \right). \end{aligned}\tag{3.65}$$

The right-hand side of (3.65) can be estimated term by term as follows. For $\|I_1^{n+1}\|_{L^4}$, we have

$$\begin{aligned}\|I_1^{n+1}\|_{L^4}^2 \le& C(\|e_\sigma^{n+1}\|_{L^4}^2 + \|\mathbf{e}_\mathbf{u}^n\|_{L^4}^2 + \|\nabla \mathbf{e}_\mathbf{u}^{n+1}\|_{L^4}^2)\\ \le& C(\|e_\sigma^{n+1}\|_{H^1}^2 + \|\nabla \mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + \|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2), \end{aligned}\tag{3.66}$$

where we use

$$\|\mathbf{u}^n\|_{L^\infty} \le C\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2} + C\|\mathbf{A}\mathbf{u}(t_n)\|_{L^2} \le C.$$

For $\|I_2^{n+1}\|_{L^4}$, we can get

$$\begin{aligned}\|I_2^{n+1}\|_{L^4}^2 \le& C\|e_\sigma^{n+1} D_\tau(\sigma(t_{n+1})\mathbf{u}(t_{n+1}))\|_{L^4}^2 + C\|\sigma^{n+1} e_\sigma^{n+1} D_\tau \mathbf{u}(t_{n+1})\|_{L^4}^2\\ &+ C\|\sigma^{n+1}(R_\sigma^{n+1} - \nabla(\sigma(t_{n+1}))\cdot \mathbf{e}_\mathbf{u}^n - \nabla e_\sigma^{n+1}\cdot \mathbf{u}^n)\mathbf{u}(t_n)\|_{L^4}^2\\ \le& C(\|e_\sigma^{n+1}\|_{H^1}^2 + \|\nabla \mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + \|\nabla \mathbf{e}_\sigma^{n+1}\|_{L^4}^2 + \|R_\sigma^{n+1}\|_{L^4}^2). \end{aligned}\tag{3.67}$$

For $\|I_3^{n+1}\|_{L^4}$, we have

$$\begin{aligned}\|I_3^{n+1}\|_{L^4}^2 \le& C(\|\nabla e_\rho^{n+1}\|_{L^4}^2 + \|\nabla\rho^{n+1}\cdot \mathbf{e}_\mathbf{u}^n\|_{L^4}^2 + \|\mathbf{e}_\mathbf{u}^{n+1}\nabla\rho^{n+1}\cdot \mathbf{u}^n\|_{L^4}^2)\\ \le& C(\|\nabla e_\sigma^{n+1}\|_{L^4}^2 + \|\nabla\rho^{n+1}\|_{L^4}^2\|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + \|\nabla\rho^{n+1}\|_{L^4}^2\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2)\\ \le& C(\|\nabla e_\sigma^{n+1}\|_{L^4}^2 + \|\mathbf{A}\mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + \|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}^2), \end{aligned}\tag{3.68}$$

where we use

$$\|\nabla\rho^{n+1}\|_{L^4}=\|2\sigma^{n+1}\nabla\sigma^{n+1}\|_{L^4}\le C\|\nabla e_\sigma^{n+1}\|_{L^4}+C\|\nabla\sigma(t_{n+1})\|_{L^4}\le C.$$

For $\|I_4^{n+1}\|_{L^4}$ and $\|I_6^{n+1}\|_{L^4}$, it is easy to prove that

$$\begin{aligned}\|I_4^{n+1}\|_{L^4}^2\le& C(\|\mathbf{e_b}^n\|_{L^4}^2+\|\mathbf{curl}\ \mathbf{e_b}^{n+1}\|_{L^4}^2)\\ \le& C(\|\mathbf{curl}\ \mathbf{e_b}^n\|_{L^2}^2+\|\mathbf{e_b}^{n+1}\|_{H^2}^2)\end{aligned}\tag{3.69}$$

and

$$\|I_6^{n+1}\|_{L^4}^2\le C\|\nabla\sigma^{n+1}\|_{L^4}^2\|\mathbf{e_u}^n\|_{H^2}^2\le C\|\mathbf{e_u}^n\|_{H^2}^2.\tag{3.70}$$

Then substituting the above estimates and (3.61) into (3.65), we have

$$\begin{aligned}&\tau\sum_{n=0}^{N-1}(\|D_\tau\mathbf{e_u}^{n+1}\|_{L^4}^2+\|\mathbf{A}\mathbf{e_u}^{n+1}\|_{L^4}^2)\\ \le& C\tau\sum_{n=0}^{N-1}(\|\nabla\phi^{n+1}\|_{L^4}^2+\|e_\sigma^{n+1}\|_{H^1}^2+\|\nabla e_\sigma^{n+1}\|_{L^4}^2+\|\mathbf{e_u}^n\|_{H^2}^2+\|\mathbf{e_u}^{n+1}\|_{H^2}^2)\\ &+C\tau\sum_{n=0}^{N-1}(\|\mathbf{curl}\ \mathbf{e_b}^n\|_{L^2}^2+\|\mathbf{e_b}^{n+1}\|_{H^2}^2+\|R_\sigma^{n+1}\|_{L^4}^2+\|\mathbf{R_u}^{n+1}\|_{L^4}^2)\le C\tau^{\frac{7}{6}},\end{aligned}$$

which implies that

$$\tau\sum_{n=0}^{N-1}\|\mathbf{e_u}^{n+1}\|_{W^{1,\infty}}^2\le C\tau\sum_{n=0}^{N-1}\|\mathbf{A}\mathbf{e_u}^{n+1}\|_{L^4}^2\le C\tau^{\frac{7}{6}}.\tag{3.71}$$

As the consequence of (3.71), we obtain the uniform boundness of $\mathbf{u}^{n+1}$ in $W^{1,\infty}$-norm:

$$\|\mathbf{u}^{n+1}\|_{W^{1,\infty}}\le\|\mathbf{e_u}^{n+1}\|_{W^{1,\infty}}+\|\mathbf{u}(t_{n+1})\|_{W^{1,\infty}}\le C,\quad\forall 0\le n\le N-1.\tag{3.72}$$

Step VI: H^2 **and** $W^{1,\infty}$ **estimates of** e_σ^{n+1}**.** Firstly, we give the estimate of $\|\nabla e_\sigma^{n+1}\|_{L^\infty}$. Testing (3.42) by $\tau|D_je_\sigma^{n+1}|^{k-1}D_je_\sigma^{n+1}$ and taking $k\to+\infty$, we have

$$\begin{aligned}&|D_je_\sigma^{n+1}\|_{L^\infty}-\|D_je_\sigma^n\|_{L^\infty}\\ \le& C\tau\|\nabla(D_j\sigma(t_{n+1}))\cdot\mathbf{e_u}^n\|_{L^\infty}+C\tau\|\nabla\sigma(t_{n+1})\cdot D_j\mathbf{e_u}^n\|_{L^\infty}\\ &+C\tau\|\nabla e_\sigma^{n+1}\cdot D_j\mathbf{u}^n\|_{L^\infty}+C\tau\|D_jR_\sigma^{n+1}\|_{L^\infty}\\ \le& C\tau\|\mathbf{e_u}^n\|_{L^\infty}+C\tau\|\nabla\mathbf{e_u}^n\|_{L^\infty}+C\tau\|\nabla e_\sigma^{n+1}\|_{L^\infty}+C\tau\|\nabla R_\sigma^{n+1}\|_{L^\infty}.\end{aligned}$$

Taking the sum of the above inequality and using the discrete Gronwall inequality for the sufficiently small τ, we get

$$\|\nabla e_\sigma^{n+1}\|_{L^\infty}\le C\tau\sum_{n=0}^{m}(\|\mathbf{e_u}^n\|_{L^\infty}+\|\nabla\mathbf{e_u}^n\|_{L^\infty}+\|\nabla R_\sigma^{n+1}\|_{L^\infty})\le C\tau^{\frac{1}{2}}\tag{3.73}$$

for all $0\le n\le N-1$.

To get the estimate $\|e_\sigma^{n+1}\|_{H^2}$, differentiating (3.42) with respect x_i yields

$$\begin{aligned}&\frac{D_{ij}e_\sigma^{n+1}-D_{ij}e_\sigma^n}{\tau}+\nabla(D_{ij}\sigma(t_{n+1}))\cdot\mathbf{e}_\mathbf{u}^n+\nabla(D_j\sigma(t_{n+1}))\cdot D_i\mathbf{e}_\mathbf{u}^n\\&\quad+\nabla(D_i\sigma(t_{n+1}))\cdot D_j\mathbf{e}_\mathbf{u}^n+\nabla\sigma(t_{n+1})\cdot D_{ij}\mathbf{e}_\mathbf{u}^n+\nabla(D_{ij}e_\sigma^{n+1})\cdot\mathbf{u}^n\\&\quad+\nabla(D_je_\sigma^{n+1})\cdot D_i\mathbf{u}^n+\nabla(D_ie_\sigma^{n+1})\cdot D_j\mathbf{u}^n+\nabla e_\sigma^{n+1}\cdot D_{ij}\mathbf{u}^n=D_{ij}R_\sigma^{n+1}\end{aligned}\tag{3.74}$$

for $i,j=1,2,3$. Multiplying the above equation by $2\tau D_{ij}\mathbf{e}_\sigma^{n+1}$ and integrating over Ω leads to

$$\begin{aligned}&\|D_{ij}e_\sigma^{n+1}\|_{L^2}^2-\|D_{ij}e_\sigma^n\|_{L^2}^2+\|D_{ij}e_\sigma^{n+1}-D_{ij}e_\sigma^n\|_{L^2}^2\\\le&C\tau(\|\mathbf{e}_\mathbf{u}^n\|_{L^\infty}^2+\|D_i\mathbf{e}_\mathbf{u}^n\|_{L^3}^2+\|D_j\mathbf{e}_\mathbf{u}^n\|_{L^3}^2+\|D_{ij}\mathbf{e}_\mathbf{u}^n\|_{L^2}^2+\|\nabla(D_ie_\sigma^{n+1})\|_{L^2}^2)\\&+C\tau(\|\nabla(D_je_\sigma^{n+1})\|_{L^2}^2+\|\nabla e_\sigma^{n+1}\|_{L^\infty}^2\|D_{ij}\mathbf{u}^n\|_{L^2}^2+\|D_{ij}R_\sigma^{n+1}\|_{L^2}^2)+C\tau\|D_{ij}e_\sigma^{n+1}\|_{L^2}^2.\end{aligned}$$

Summing up the above inequality and using the discrete Gronwall inequality gives

$$\|\nabla^2e_\sigma^{n+1}\|_{L^2}^2\le C\tau\sum_{i=0}^n\left(\|\mathbf{A}\mathbf{e}_\mathbf{u}^i\|_{L^2}^2+\|\nabla e_\sigma^{i+1}\|_{L^\infty}^2\|\mathbf{A}\mathbf{u}^i\|_{L^2}^2+\|\nabla^2R_\sigma^{i+1}\|_{L^2}^2\right)\le C\tau^{1/4},$$

which implies that

$$\|e_\sigma^{n+1}\|_{H^2}\le C\tau^{1/8},\quad\forall 0\le n\le N-1.\tag{3.75}$$

Step VII: The close of mathematical induction. To finish the proof of Theorem 3.1, we need to close the mathematical induction, i.e., we need to prove that (3.3) is valid for $k=n+1$ with $0\le n\le N-1$. In terms of error estimates (3.49), (3.71), (3.73) and (3.75), we can see

$$\begin{aligned}&\|\mathbf{u}^{n+1}\|_{W^{1,\infty}}+\|\mathbf{A}\mathbf{u}^{n+1}\|_{L^2}+\|\sigma^{n+1}\|_{W^{1,\infty}}+\|\sigma^{n+1}\|_{H^2}\\\le&\|\mathbf{e}_\mathbf{u}^{n+1}\|_{W^{1,\infty}}+\|\mathbf{A}\mathbf{e}_\mathbf{u}^{n+1}\|_{L^2}+\|e_\sigma^{n+1}\|_{W^{1,\infty}}+\|e_\sigma^{n+1}\|_{H^2}+(K-1)\\\le&C\tau^{1/12}+(K-1)\le K\end{aligned}$$

for some small $\tau\le\tau_0$ with $C\tau_0^{1/12}\le 1$. Error estimates (3.4a) and (3.4b) follow from (3.21), (3.25), (3.48), (3.49), (3.52) and (3.54). Thus, we complete the proof of Theorem 3.1. □

4 Numerical experiment

In this section, for the reason of simplicity, we make the numerical experiment for the 2D MHD system with variable density to support the stability of the numerical solution and the temporal convergence rate $\mathcal{O}(\tau)$ derived in Theorem 3.1 by using the finite element method to discretize (2.8)-(2.10), where the mini-element (P_1b-P_1) is used to approximate $(\mathbf{u},p)$, and the P_1 element is used to approximate σ and $\mathbf{b}$. The associated vector-valued finite element spaces are denoted by $\mathbf{V}_h\subset\mathbf{V}$, $M_h\subset M$, $X_h\subset H^1(\Omega)$ and $\mathbf{W}_h\subset\mathbf{W}$, respectively. It is clear that $\mathbf{V}_h\cdot\mathbf{V}_h\subset X_h$ does not hold.

We solve the incompressible MHD equations (2.5a)-(2.6) with variable density in the unit square:

$$\Omega=\{(x,y)\in\mathbf{R}^2:\ 0<x<1,\ 0<y<1\}.$$

In the case of two-dimensional domain, the MHD system (2.5a)-(2.5d) with variable density are

$$\sigma_t+\nabla\sigma\cdot\mathbf{u}=0, \tag{4.1a}$$

$$\sigma(\sigma\mathbf{u})_t-\frac{1}{Re}\Delta\mathbf{u}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}+\frac{\mathbf{u}}{2}\nabla\cdot(\rho\mathbf{u})+S\mathbf{b}\times curl\mathbf{b}+\nabla p=\mathbf{f}, \tag{4.1b}$$

$$\mathbf{b}_t+\frac{1}{Rm}\mathbf{curl}\ (curl\ \mathbf{b})-\mathbf{curl}\ (\mathbf{u}\times\mathbf{b})=\mathbf{g}, \tag{4.1c}$$

$$\nabla\cdot\mathbf{u}=0,\quad \nabla\cdot\mathbf{b}=0. \tag{4.1d}$$

Here $\mathbf{u}=(u_1,u_2)$ and $\mathbf{b}=(b_1,b_2)$. In the 2D case, the operator *curl* applied to a vector $\mathbf{v}=(v_1,v_2)$ is defined by

$$curl\mathbf{v}=\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}.$$

The operator **curl** applied to a scalar function r is defined by

$$\mathbf{curl}\ r=\left(\frac{\partial r}{\partial y},-\frac{\partial r}{\partial x}\right).$$

In addition, the cross product of two vectors $\mathbf{u}$ and $\mathbf{b}$ is given by $\mathbf{u}\times\mathbf{b}=u_1b_2-u_2b_1$. For a vector function $\mathbf{b}$ and a scalar function r, the cross product is given by $\mathbf{b}\times r=(rb_2,-rb_1)$.

Let $\sigma_h^0=\pi_h\sigma_0$, $\mathbf{u}_h^0=I_h\mathbf{u}_0$ and $\mathbf{b}_h^0=J_h\mathbf{b}_0$, where π_h, I_h and J_h are L^2 or $\mathbf{L}^2$ projection operators onto the finite element spaces. For $0\le n\le N-1$, the finite element fully discrete scheme can be stated as follows:

Step I: For given $\sigma_h^n\in W_h$ and $\mathbf{u}_h^n\in\mathbf{V}_h$, we solve $\sigma_h^{n+1}\in W_h$ such that

$$(D_\tau\sigma_h^{n+1},r_h)+(\nabla\sigma_h^{n+1}\cdot\mathbf{u}_h^n,r_h)+\frac{1}{2}(\sigma_h^{n+1}\nabla\cdot\mathbf{u}_h^n,r_h)=0,\quad \forall r_h\in W_h. \tag{4.2}$$

Step II: For given $\mathbf{u}_h^n\in\mathbf{V}_h$ and $\mathbf{b}_h^n\in\mathbf{W}_h$, we solve $(\mathbf{u}_h^{n+1},p_h^{n+1})\in\mathbf{V}_h\times M_h$ and $\mathbf{b}_h^{n+1}\in\mathbf{W}_h$ such that

$$\begin{aligned}&\sigma_h^{n+1}\left(D_\tau(\sigma_h^{n+1}\mathbf{u}_h^{n+1}),\mathbf{v}_h\right)+\frac{1}{Re}(\nabla\mathbf{u}_h^{n+1},\nabla\mathbf{v}_h)-(\nabla\cdot\mathbf{v}_h,p_h^{n+1})+(\nabla\cdot\mathbf{u}_h^{n+1},q_h)\\&\quad+\left(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^{n+1},\mathbf{v}_h\right)+\frac{1}{2}\left(\mathbf{u}_h^{n+1}\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n),\mathbf{v}_h\right)+S(\mathbf{b}_h^n\times curl\ \mathbf{b}_h^{n+1},\mathbf{v}_h)\\&=(\mathbf{f}^{n+1},\mathbf{v}_h)\end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
&(D_\tau \mathbf{b}_h^{n+1},\mathbf{w}_h)+\frac{1}{Rm}(curl\ \mathbf{b}_h^{n+1},curl\ \mathbf{w}_h)+\frac{1}{Rm}(\nabla\cdot\mathbf{b}_h^{n+1},\nabla\cdot\mathbf{w}_h)\\
&\quad-(\mathbf{u}_h^{n+1}\times\mathbf{b}_h^{n},curl\ \mathbf{w}_h)=(\mathbf{g}^{n+1},\mathbf{w}_h)
\end{aligned}\tag{4.4}
$$

for any $(\mathbf{v}_h,q_h)\in\mathbf{V}_h\times M_h$ and $\mathbf{w}_h\in\mathbf{W}_h$, where $\rho_h^{n+1}=(\sigma_h^{n+1})^2$.

Remark 4.1. Due to $\nabla\cdot\mathbf{u}_h^n\neq 0$ in the point-wise sense, the stable term $\frac{1}{2}(\sigma_h^{n+1}\nabla\cdot\mathbf{u}_h^n,r_h)$ in (4.2) is added to preserve the unconditional stability of σ_h^{n+1}.

For the finite element fully discrete scheme (4.2)-(4.4), we have the following unconditional stability which implies the existence and uniqueness of the numerical approximation solution $(\sigma_h^{n+1},\mathbf{u}_h^{n+1},p_h^{n+1},\mathbf{b}_h^{n+1})$ for $0\le n\le N-1$.

Theorem 4.1. *For any time step size τ and mesh size h, there hold the following discrete energy inequalities:*

$$
\|\sigma_h^{n+1}\|_{L^2}\le\|\sigma_h^0\|_{L^2}\tag{4.5}
$$

and

$$
\begin{aligned}
&\|\sigma_h^{n+1}\boldsymbol{u}_h^{n+1}\|_{L^2}^2+S\|\boldsymbol{b}_h^{n+1}\|_{L^2}^2+\frac{\tau}{Re}\sum_{i=0}^{n}\|\nabla\boldsymbol{u}_h^{i+1}\|_{L^2}^2\\
&\qquad+\frac{S\tau}{Rm}\sum_{i=0}^{n}\Big(\|curl\ \boldsymbol{b}_h^{i+1}\|_{L^2}^2+\|\nabla\cdot\boldsymbol{b}_h^{i+1}\|_{L^2}^2\Big)\\
&\le\|\sigma_h^0\boldsymbol{u}_h^0\|_{L^2}^2+S\|\boldsymbol{b}_h^0\|_{L^2}^2+C\tau\sum_{i=0}^{N-1}\Big(\|\boldsymbol{f}^{i+1}\|_{L^2}^2+\|\boldsymbol{g}^{i+1}\|_{L^2}^2\Big)
\end{aligned}\tag{4.6}
$$

for all $0\le n\le N-1$.

Proof. In fact, setting $r_h=\sigma_h^{n+1}$ in (4.2), we have

$$
\|\sigma_h^{n+1}\|_{L^2}^2+\|\sigma_h^{n+1}-\sigma_h^n\|_{L^2}^2=\|\sigma_h^n\|_{L^2}^2
$$

by using

$$
(\nabla\sigma_h^{n+1}\cdot\mathbf{u}_h^n,\sigma_h^{n+1})=\frac{1}{2}\int_\Omega\nabla|\sigma_h^{n+1}|^2\cdot\mathbf{u}_h^n dx=-\frac{1}{2}\int_\Omega|\sigma_h^{n+1}|^2(\nabla\cdot\mathbf{u}_h^n)dx.
$$

Thus, the stability (4.5) of σ_h^{n+1} holds.

Setting $(\mathbf{v}_h,q_h)=2\tau(\mathbf{u}_h^{n+1},p_h^{n+1})$ in (4.3) and $\mathbf{w}_h=2S\tau\mathbf{b}_h^{n+1}$ in (4.4), and adding the resulting equations, we have

$$
\begin{aligned}
&\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2-\|\sigma_h^n\mathbf{u}_h^n\|_{L^2}^2+\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}-\sigma_h^n\mathbf{u}_h^n\|_{L^2}^2\\
&\quad+S\|\mathbf{b}_h^{n+1}\|_{L^2}^2-S\|\mathbf{b}_h^n\|_{L^2}^2+S\|\mathbf{b}_h^{n+1}-\mathbf{b}_h^n\|_{L^2}^2+\frac{2\tau}{Re}\|\nabla\mathbf{u}_h^{n+1}\|_{L^2}^2\\
&\quad+\frac{2S\tau}{Rm}\Big(\|curl\ \mathbf{b}_h^{n+1}\|_{L^2}^2+\|\nabla\cdot\mathbf{b}_h^{n+1}\|_{L^2}^2\Big)\\
&=2\tau(\mathbf{f}^{n+1},\mathbf{u}_h^{n+1})+2\tau(\mathbf{g}^{n+1},\mathbf{b}_h^{n+1})
\end{aligned}
$$

by using (2.14) and

$$\left(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1}\right)=\frac{1}{2}\int_\Omega \rho_h^{n+1}\mathbf{u}_h^n\cdot\nabla|\mathbf{u}_h^{n+1}|^2dx=-\frac{1}{2}\int_\Omega \nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n)|\mathbf{u}_h^{n+1}|^2dx.$$

By using the Hölder and Young inequalities, we can get the stability (4.5) of $(\mathbf{u}_h^{n+1},\mathbf{b}_h^{n+1})$. Thus, we finish the proof of Theorem 4.1. □

Now, we present numerical results to confirm the stability of the numerical solution and the temporal convergence rate $\mathcal{O}(\tau)$ derived in Theorem 3.1. We take the appropriate $\mathbf{f}$ and $\mathbf{g}$ in (4.1b)-(4.1c) such that the exact solutions $(\sigma,\mathbf{u},p,\mathbf{b})$ have the following forms:

$$\begin{aligned}
&\sigma(x,y,t)=\sqrt{2+x\cos(\sin(t))+y\sin(\sin(t))},\\
&\mathbf{u}(x,y,t)=(-y\cos(t),\ x\cos(t))^{\mathrm{T}},\\
&p(x,y,t)=\sin(x)\sin(y)\sin(t),\\
&\mathbf{b}(x,y,t)=(-y\sin(t),\ x\sin(t))^{\mathrm{T}}.
\end{aligned}$$

In addition, we take the Reynolds number $Re=100$, the magnetic Reynolds number $Rm=100$, the coupling number $S=1$, the mesh size $h=\frac{1}{200}$ and the final time $T=1$. For brevity of notations, we denote

$$\begin{aligned}
&\|\rho-\rho_h\|_{L^2}=\left\|\rho(t_N)-\rho_h^N\right\|_{L^2},\\
&\|\sigma\mathbf{u}-\sigma_h\mathbf{u}_h\|_{L^2}=\left\|\sigma(t_N)\mathbf{u}(t_N)-\sigma_h^N\mathbf{u}_h^N\right\|_{L^2},\\
&\|\mathbf{u}-\mathbf{u}_h\|_V=\left\|\nabla\left(\mathbf{u}(t_N)-\mathbf{u}_h^N\right)\right\|_{L^2},\\
&\|\mathbf{b}-\mathbf{b}_h\|_{L^2}=\left\|\mathbf{b}(t_N)-\mathbf{b}_h^N\right\|_{L^2},\\
&\|\mathbf{b}-\mathbf{b}_h\|_W=\left\|curl\left(\mathbf{b}(t_N)-\mathbf{b}_h^N\right)\right\|_{L^2}+\left\|\nabla\cdot\left(\mathbf{b}(t_N)-\mathbf{b}_h^N\right)\right\|_{L^2}.
\end{aligned}$$

For some Banach space X with norm $\|\cdot\|_X$, we define the discrete $l^2(X)$-norm by

$$\begin{aligned}
&\|\mathbf{v}\|_{l^2(X)}=\left(\tau\sum_{n=1}^N\|\mathbf{v}(t_n)\|_X^2\right)^{1/2},\\
&\|\mathbf{v}_h\|_{l^2(X)}=\left(\tau\sum_{n=1}^N\|\mathbf{v}_h^n\|_X^2\right)^{1/2},
\end{aligned}$$

for any $\mathbf{v}(t)\in X$ and $\mathbf{v}_h:=\{\mathbf{v}_h^n\}_{n=1}^N\subset X$.

To confirm the stability of the numerical solution and the first-order temporal convergence rate, the time step τ is taken $\tau=\frac{1}{5\times 2^i}$ for $i=0,1,\cdots,4$. Numerical results are displayed in the following tables. In Tables 1 and 2, the stabilities of the exact solution

Table 1: Stabilities of the exact solution $(\sigma,\mathbf{u},\mathbf{b})$.

τ	$\|\sigma(t_N)\|_{L^2}$	$\|\sigma(t_N)\mathbf{u}(t_N)\|_{L^2}$	$\|\mathbf{u}\|_{l^2(V)}$	$\|\mathbf{b}\|_{l^2(W)}$
0.2	1.92510	0.86925	0.34172	0.75263
0.1	1.92510	0.86925	0.24163	0.53219
0.05	1.92510	0.86925	0.17086	0.37632
0.025	1.92510	0.86925	0.12082	0.26610
0.0125	1.92510	0.86925	0.08543	0.18816

Table 2: Stabilities of the numerical solution $(\sigma_h,\mathbf{u}_h,\mathbf{b}_h)$.

τ	$\|\sigma_h^N\|_{L^2}$	$\|\sigma_h^N\mathbf{u}_h^N\|_{L^2}$	$\|\mathbf{u}_h\|_{l^2(V)}$	$\|\mathbf{b}_h\|_{l^2(W)}$
0.2	1.91748	0.85456	0.36269	0.76021
0.1	1.92134	0.86142	0.24539	0.53364
0.05	1.92323	0.86519	0.17153	0.37658
0.025	1.92417	0.86718	0.12093	0.26614
0.0125	1.92463	0.86821	0.08545	0.18817

Table 3: Numerical errors I and temporal convergence rates.

τ	$\|\rho-\rho_h\|_{L^2}$	$\|\sigma\mathbf{u}-\sigma_h\mathbf{u}_h\|_{L^2}$	$\|\mathbf{b}-\mathbf{b}_h\|_{L^2}$
0.2	3.80734e-002	3.02732e-002	3.30688e-002
0.1	1.89736e-002	1.61293e-002	1.67111e-002
0.05	9.50156e-003	8.36358e-003	8.40923e-003
0.025	4.76004e-003	4.26555e-003	4.21981e-003
0.0125	2.38386e-003	2.15517e-003	2.11394e-003
rate	0.999	0.953	0.992

Table 4: Numerical errors II and temporal convergence rates.

τ	$\|\mathbf{u}-\mathbf{u}_h\|_V$	$\|\mathbf{b}-\mathbf{b}_h\|_W$	$\|\nabla(p-p_h)\|_{l^2(L^2)}$
0.2	2.71756e-001	2.39326e-001	6.78071e-001
0.1	1.35306e-001	1.24142e-001	2.03986e-001
0.05	6.77424e-002	6.33905e-002	6.54700e-002
0.025	3.39541e-002	3.20538e-002	2.19734e-002
0.0125	1.70059e-002	1.61194e-002	7.56432e-003
rate	1.000	0.973	1.621

$(\sigma,\mathbf{u},\mathbf{b})$ and the numerical solution $(\sigma_h,\mathbf{u}_h,\mathbf{b}_h)$ are presented. From Table 2, we can see the good stability of the numerical solution of (4.2)-(4.3) when the time step τ becomes small and small gradually. Numerical errors of $(\rho_h,\mathbf{u}_h,p_h,\mathbf{b}_h)$ are shown in Table 3, from which we can see clearly the predict first-order convergence rate $\mathcal{O}(\tau)$. In summary, the above numerical results are in good agreement with our theoretical analysis.

5 Conclusions

Based on an equivalent form of the incompressible MHD system with variable density, we proposed a first-order Euler time-discrete scheme for approximating the equivalent system. In the finite element approximations, we do not need to consider the constraint condition (1.6) and the finite element algorithm is unconditionally stable. In terms of the discrete maximal L^p-regularity of the Stokes problem, we proved the first-order convergence rate $\mathcal{O}(\tau)$ for the proposed time-discrete scheme. On the other hand, for the equivalent system (2.5a)-(2.5d), it is easy to construct the unconditionally stable BDF2 scheme without any conditions of the time step size and mesh size, which will be given in future work.

Acknowledgements

This work was supported by National Natural Science Foundation of China (No. 11771337) and Natural Science Foundation of Zhejiang Province (No. LY23A010002).

References

[1] H. Abidi and M. Paicu, *Global existence for the magnetohydrodynamic system in critical spaces,* Proc. Roy. Soc. Edinburgh Sect. A, 138 (2008), pp. 447–476.

[2] R. An, *Error analysis of a new fractional-step method for the incompressible Navier-Stokes equations with variable density,* J. Sci. Comput., 84 (2020), p. 3.

[3] R. An, *Iteration penalty method for the incompressible Navier-Stokes equations with variable density based on the artificial compressible method,* Adv. Comput. Math., 46 (2020), p. 5.

[4] R. An, *Error analysis of a time-splitting method for incompressible flows with variable density,* Appl. Numer. Math., 150 (2020), pp. 384–395.

[5] R. An and Y. Li, *Error analysis of first-order projection method for time-dependent magnetohydrodynamics equations,* Appl. Numer. Math., 112 (2017), pp. 167–181.

[6] R. An and C. Zhou, *Error analysis of a fractional-step method for magnetohydrodynamics equations,* J. Comput. Appl. Math., 313 (2017), pp. 168–184.

[7] Q. Y. Bie, Q. R. Wang and Z. A. Yao, *Global well-posedness of the 3D incompressible MHD equations with variable density,* Nonlinear Anal. Real World Appl., 47 (2019), pp. 85–105.

[8] H. Cabannes, Theoretical Magnetofluiddynamics, Academic Press, New York, 1970.

[9] W. T. Cai, B. Y. Li and Y. Li, *Error analysis of a fully discrete finite element method for variable density incompressible flows in two dimensions,* ESAIM: Math. Model. Numer. Anal., 55 (2021), pp. S103–S147.

[10] H. T. Chen, J. J. Mao and J. Shen, *Error estimate of Gauge-Uzawa methods for incompressible flows with variable density,* J. Comput. Appl. Math., 364 (2020), 112321.

[11] Q. Chen, Z. Tan and Y. J. Wang, *Strong solutions to the incompressible magnetohydrodynamic equations,* Math. Methods Appl. Sci., 34 (2011), pp. 94–107.

[12] B. Desjardins and C. Le Bris, *Remarks on a nonhomogeneous model of magnetohydrodynamics,* Differential Ingegral Equations, 11 (1998), pp. 377–394.

[13] G. P. GALDI, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Springer-Verlag, New York, 1994.

[14] H. D. GAO AND W. F. QIU, *A semi-implict energy conserving finite element method for the dynamical incompressible magnetohydrodynamics equations,* Comput. Methods Appl. Mech. Eng., 346 (2019), pp. 982–1001.

[15] J. GERBEAU AND C. LE BRIS, *Existence of solution for a density-dependent magnetohydrodynamic equation,* Adv. Differential Equations, 2 (1997), pp. 427–452.

[16] J. GERBEAU, C.LE BRIS AND T. LELIÈVRE, Mathematical Methods for the Magnetohydrodynamics of Liquid Metals, Oxford: Oxford University Press, 2006.

[17] F. GUILLÉN-GONZÁLEZ AND J. V. GUTIÉRREZ-SANTACREU, *unconditional stability and convergence of fully discrete schemes for 2D viscous fluids models with mass diffusion,* Math. Comput., 2008 (77), pp. 1495–1524.

[18] F. GUILLÉN-GONZÁLEZ AND J.V. GUTIÉRREZ-SANTACREU, *Conditional stability and convergence of a fully discrete scheme for three-dimensional Navier-Stokes equations with mass diffusion,* SIAM J. Numer. Math., 2008 (46), pp. 2276–2308.

[19] F. GUILLÉN-GONZÁLEZ AND J. V. GUTIÉRREZ-SANTACREU, *Error estimates of a linear decoupled Euler-FEM scheme for a mass diffusion,* Numer. Math., 2011 (117), pp. 333–371.

[20] J. L. GUERMOND AND L. QUARTAPELLE, *A projection FEM for variable density incomressible flows,* J. Comput. Phys., 165 (2000), pp. 167–188.

[21] J. L. GUERMOND AND A. SALGADO, *A splitting method for incompressible flows with variable density based on a pressure Poisson equation,* J. Comput. Phys., 228 (2009), pp. 2834–2846.

[22] J. L. GUERMOND AND A. SALGADO, *Error analysis of a fractional time-stepping technique for incompressible flows with variable density,* SIAM J. Numer. Anal., 49 (2011), pp. 917–944.

[23] G. L. GUI, *Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity,* J. Func. Anal., 267 (2014), pp. 1488–1539.

[24] Y. N. HE, *Unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equations,* IMA J. Numer. Anal., 35 (2015), pp. 767–801.

[25] J. HEYWOOD AND R. RANNACHER, *Finite-element approximation of the nonstationary Navier-Stokes problem Part IV: error analysis for second-order time discretization,* SIAM J. Numer. Anal., 27 (1990), pp. 353–384.

[26] X. D. HUANG AND Y. WANG, *Global strong solution to the 2D nonhomogeneous incompressible MHD system,* J. Differential Equations, 254 (2013), pp. 511–527.

[27] W. HUGHES AND F. YOUNG, The Electromagnetics of Fluids, Wiley, New York, 1966.

[28] B. KOVÁCS, B. Y. LI AND C. LUBICH, *A-stable time discretizations preserve maximal parabolic regularity,* SIAM J. Numer. Anal., 54 (2016), pp. 3600–3624.

[29] B. Y. LI, H. D. GAO AND W. W. SUN, *Unconditionally optimal error estimates of a Crank-Nicolson Galerkin method for the nonlinear thermistor equations,* SIAM J. Numer. Anal., 52 (2014), pp. 933–954.

[30] B. Y. LI, W. F. QIU AND Z. Z. YANG, *A convergent post-processed discontinuous Galerkin method for incompressible flow with variable density,* J. Sci. Comput., 91 (2022), p. 2.

[31] B. Y. LI AND W. W. SUN, *Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations,* Int. J. Numer. Anal. Model., 10 (2013), pp. 622–633.

[32] B. Y. LI AND W. W. SUN, *Unconditional convergence and optimal error estimates of a Galerkin-mixed FEM for incompressible miscible flow in porous media,* SIAM J. Numer. Anal., 51 (2013), pp. 1959–1977.

[33] B. Y. LI, J. L. WANG AND L. W. XU, *A convergent linearized Lagrange finite element method for the magnetohydrodynamic equations in two-dimensional nonsmooth and nonconvex domains,* SIAM

J. Numer. Anal., 58 (2020), pp. 430–459.

[34] X. L. Li and D. H. Wang, *Global strong solution to the three-dimensional density-dependent incompressible magnetohydrodynamic flows,* J. Differential Equations, 251 (2011), pp. 1580–1615.

[35] Y. Li, L. Q. Mei, J. T. Ge and F. Shi, *A new fractional time-stepping method for variable density incompressible flows,* J. Comput. Phys, 242 (2013), pp. 124–137.

[36] Y. Li and R. An, *Temporal error analysis of Euler semi-implicit scheme for the magnetohydrodynamics equations with variable density,* Appl. Numer. Math., 166 (2021), pp. 146–167.

[37] Y. Li and X. L. Luo, *Second-order semi-implicit Crank-Nicolson scheme for a coupled magnetohydrodynamics system,* Appl. Numer. Math., 145 (2019), pp. 48–68.

[38] Y. Li, Y. J. Ma and R. An, *Decoupled, semi-implicit scheme for a coupled system arising in magnetohydrodynamics problem,* Appl. Numer. Math., 127 (2018), pp. 142–163.

[39] R. Moreau, Magnetohydrodynamics, Kluwer Academic Publishers, 1990.

[40] A. Prohl, *Convergent finite element discretizations of the nonstationary incompressible magnetohydrodynamic system,* ESAIM:M2AN, 42 (2008), pp. 1065–1087.

[41] J. H. Pyo and J. Shen, *Gauge-Uzawa methods for incomressible flows with variable density,* J. Comput. Phys., 221 (2007), pp. 181–197.

[42] X. F. Yang, G. D. Zhang and X. M. He, *Convergence analysis of an Unconditionally energy stable projection scheme for magnetohydrodynamic equations,* Appl. Numer. Math., 136 (2019), pp. 235–256.

[43] G. D. Zhang, X. M. He and X. F. Yang, *A decoupled, linear and unconditionally energy stable scheme with finite element discretizations for magnetohydrodynamic equations,* J. Sci. Comput., 81 (2019), pp. 1678–1711.

[44] G. D. Zhang, X. M. He and X. F. Yang, *Fully decoupled, linear and unconditionally energy stable time discretization for solving the magnetohydrodynamic equations,* J. Comput. Appl. Math., 369 (2020), 112636.

[45] G. D. Zhang and Y. N. He, *Decoupled schemes for unsteady MHD equations II: finite element spatial discretization and numerical implementation,* Comput. Math. Appl., 69 (2015), pp. 1390–1406.

[46] Y.H. Zhang, Y.R. Hou and L. Shan, Numerical analysis of the Crank-Nicolson extrapolation time discrete scheme for magnetohydrodynamics flows, Numer. Methods Partial Differential Equations., 31 (2015), pp. 2169–2208.