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A Truncated-Type Explicit Numerical Method for the Stochastic Allen-Cahn Equation

Weijun Zhan[1] and Qian Guo[2,*]

[1] *Department of Mathematics, Anhui Normal University, Wuhu, Anhui 241000, China*
[2] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*

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Abstract. The stochastic Allen-Cahn equations, as a typical example of nonlinear stochastic partial differential equations, play an important role in phase theory. In this paper, we investigate the rate of convergence in the pth mean for a truncated-type explicit Euler time-stepping method applied to the stochastic Allen-Cahn equations as well as using the spectral Galerkin approximation in spatial discretization. Finally, a numerical example is given to confirm the strong convergence order.

AMS subject classifications: 60H35, 60H15

Key words: Stochastic Allen-Cahn equations, truncated Euler–Maruyama, spectral Galerkin method, strong convergence.

1 Introduction

Over the past decades, there are plenty of research articles about stochastic partial differential equations (SPDEs). Since it is difficult to obtain the analytical solution to a SPDE, various numerical methods have been introduced for the SPDEs with drift and diffusion coefficients satisfying the global Lipschitz or linear growth condition [1, 14, 18, 19, 25, 29]. When numerically solving a SPDE, spatial discretizations are usually achieved with finite element, finite difference, and spectral Galerkin methods; see, for example, [20,21,26,37]. The temporally discretization is usually carried out by Euler-type methods [3–5,23,25,36].

As a typical example of SPDEs, stochastic Allen-Cahn equations play an important role in the phase theory and phase transition, and this class of equations have received increasing attention in the last few years [2, 8, 23–26]. However, once the nonlinear term of the coefficient of a stochastic differential equation grows super-linearly, the standard Euler-Maruyama (EM) method is known to diverge in the strong sense [17]. Thus, the implicit scheme [9, 30], modified EM method [5, 10], splitting scheme [6, 7], and other

*Corresponding author.
Email: qguo@shnu.edu.cn (Q. Guo)

numerical methods [11, 28] have been applied to the stochastic Allen-Cahn equation. In a very recent work, Wang [36] proved a tamed EM method for temporal discretization combined with the spectral Galerkin method in space can achieve strong order 1/2. On the other side, truncated-type scheme is another efficient tool to approximate a stochastic differential equation with superlinear coefficients. There are several kinds of truncated methods designed for solving stochastic differential equation [4, 15, 16, 27, 32, 33]. In this paper, to our best knowledge, we apply, for the first time, this type of truncated scheme for the temporal discretization to construct fully discrete numerical estimators for the stochastic Allen-Cahn equation.

The rest of this paper is organized as follows. The next section presents assumptions and notations that are used throughout this paper. Section 3 is devoted to the analysis of the strong convergence rate for the full space-time discretization method to solve the underlying stochastic Allen-Cahn equation. Numerical experiments are illustrated in Section 4 to verify the theoretical findings.

2 Notations and assumptions

In this paper, we are interested in stochastic Allen-Cahn equations with additive space-time white noise, and it is described by

$$\begin{cases} \mathrm{d}X(t)+AX(t)\mathrm{d}t=F(X(t))\mathrm{d}t+\mathrm{d}W(t), & t\in(0,T], \\ X(0)=X_0. \end{cases} \tag{2.1}$$

Here the linear operator A, deterministic mappings F and I-cylindrical Wiener process $W(t,\cdot)$ that will be specified precisely later. Next, we give some notations and assumptions.

Throughout this article, unless otherwise specified, we use the following notation. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}_0$-null sets). We list some of the notation used throughout the paper as follows.

Symbol	Meaning
$(\mathbb{H},\langle\cdot,\cdot\rangle,\\|\cdot\\|)$	a real separable Hilbert space with $\\|\cdot\\|=\langle\cdot,\cdot\rangle^{\frac{1}{2}}$.
$\mathcal{L}(\mathbb{H})$	the space of bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ endowed with the usual operator norm $\\|\cdot\\|_{\mathcal{L}(\mathbb{H})}$.
$\mathcal{L}_2:=\mathcal{L}_2(\mathbb{H})\subset\mathcal{L}(\mathbb{H})$	the subspace consisting of all Hilbert-Schmidt operators from $\mathbb{H}$ to $\mathbb{H}$.
$L^\gamma(D):=L^\gamma(D;\mathbb{R})$, $\gamma\geq 1$	a Banach space consisting of γ-times integrable functions.
$V:=C(D,\mathbb{R})$	a Banach space of continuous functions with usual norms.
$\dot{\mathbb{H}}^\gamma:=\mathrm{dom}(A^{\frac{\gamma}{2}})$	the Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle_\gamma:=\langle A^{\frac{\gamma}{2}}\cdot,A^{\frac{\gamma}{2}}\cdot\rangle$ and norm $\\|\cdot\\|_\gamma=\langle\cdot,\cdot\rangle_\gamma^{\frac{1}{2}}$.
$-A$	the Laplacian with homogeneous Dirichlet boundary conditions, defined by $-Au=\Lambda u$.
λ_i, $e_i(x)$	eigenvalues and eigenvectors, here we let $\lambda_i=\pi^2 i^2$, $i\in\mathbb{N}$ and an orthonormal basis $\{e_i(x)=\sqrt{2}\sin(i\pi x),x\in(0,1)\}_{i\in\mathbb{N}}$, such that $Ae_i=\lambda_i e_i$.