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The Spatial-Temporal Fourth-Order Conservative Characteristic Runge-Kutta Finite Difference Method for Convection-Dominated Diffusion Equation

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Abstract. In this paper, we develop a new class of conservative characteristic finite difference methods for solving convection-dominated diffusion problems with fourth-order accuracy in both time and space. Specifically, the method of characteristics is utilized to handle the convection term, which allows for greater flexibility in the choice of time step sizes. To achieve high-order temporal accuracy, we propose characteristics-based optimal implicit strong stability preserving (SSP) Runge-Kutta methods implemented along the streamline. Furthermore, a conservative interpolation is employed to calculate values at the tracking points. By introducing diverse fourth-order approximation operators on the uniform Eulerian and irregular Lagrangian meshes, we can deal with the diffusion term with high accuracy while preserving the conservation property. The mass conservation for our proposed method is theoretically proved, and is verified through numerical experiments. Moreover, the numerical tests demonstrate that our scheme achieves temporal and spatial fourth-order accuracy and generates non-oscillatory solutions, even with large time step sizes.

AMS subject classifications: 65M25, 65N06

Key words: Convection-dominated diffusion equations, spatial-temporal fourth-order, mass conservation, characteristic method.

1 Introduction

Many physical phenomena can be modeled by partial differential equations, among which the convection-diffusion equation holds significant importance with a wide range of

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applications, such as fluid flowing, atmospheric environment simulation, groundwater modeling and financial computation [3, 12, 28]. In the realm of fluid dynamics, the intricate interplay between convection and diffusion plays a pivotal role in ascertaining the stability and dynamics of fluid systems [31, 32], thus convection-diffusion problems have been extensively studied over the years. Unfortunately, finding the exact solution to convection-diffusion equations is often impractical or impossible, leading to the development of reliable numerical schemes with high precision to solve the problem [12, 14, 16, 22, 23, 36, 38]. This research is crucial as it enables further exploration of physical phenomena and properties of systems governed by convection-diffusion equations.

In consideration of large computational regions and long period prediction, one of the key indicators of numerical methods is computational efficiency. When numerically solving convection-diffusion equations, particularly in cases where convection dominates diffusion, traditional schemes suffer from severe CFL restrictions and require fine time discretization with high computational costs [2, 5]. To address these issues, specific techniques have been developed to treat the convection term with care [9, 13, 16, 19, 27, 34]. The characteristics method is particularly attractive in that problems can be solved effectively along the streamline with high-order accuracy, resulting in significant computational savings. Over the years, many numerical schemes based on characteristics technique have been proposed to solve convection-diffusion equation [1, 4, 8–12, 14, 18, 23–26, 29, 35, 37, 39], which avoid the issues of non-physical oscillations and excessive numerical dissipation at steep fronts even with the coarse time step sizes [10]. In the realm of one-dimensional convection-diffusion problems, a modified method of characteristics was first proposed by Douglas and Russell in [9], which possesses first-order accuracy in time. Since then, temporal first-order characteristic methods have been further developed and applied to problems in high dimensions including aerosol transport and miscible displacement in porous media [4, 8, 18, 25, 26, 35]. However, these techniques can't guarantee mass conservation, which is a critical requirement for various mathematical modeling applications [15]. To overcome this limitation, Arbogast and Wheeler developed a characteristics-mixed finite element method conserving mass based on a space-time variational form of the convection-diffusion equation in [1]. Furthermore, temporal second-order conservative characteristic schemes have been proposed [29, 39] and further developed. [10] introduced a temporal second-order and spatial high-order scheme to deal with the aerosol convection-diffusion process. In the context of two-dimensional convection-diffusion problems, a spatial second-order conservative characteristic scheme was developed in [11], and later [12] achieved fourth-order spatial accuracy.

To enhance the accuracy of time discretization in computational simulations, numerous temporal high-order schemes have been devised. A notable category among these schemes is the strong stability preserving (SSP) temporal discretizations, which are first tailored to ensure the stability properties of the numerical solutions to hyperbolic problems [17]. These methods are known for their ability to preserve the strong stability properties of the spatial discretization coupled with the first-order Euler temporal discretiza-