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Convergence Analysis of a Global-in-Time Iterative Decoupled Algorithm for Biot's Model

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Abstract. Biot's model is a multiphysics model that describes the interaction of a poroelastic material with its interstitial fluid flow. In this study, we focus on investigating the convergence behavior of a global-in-time iterative decoupled algorithm based on a three-field formulation. During each iteration, the algorithm involves solving a reaction-diffusion subproblem across the entire temporal domain, followed by resolving a Stokes subproblem over the same time interval. This algorithm is recognized for its "partially parallel-in-time" property, enabling the implementation of a parallel procedure when addressing the Stokes subproblem. We establish its global convergence with a new technique by confirming that the limit of the sequence of numerical solutions of the global-in-time algorithm is the numerical solution of the fully coupled algorithm. Numerical experiments validate the theoretical predictions and underline the efficiency gained by implementing the parallel procedure within the proposed global-in-time algorithm.

AMS subject classifications: 65N30, 65N45, 65N15, 65Pxx

Key words: Biot's model, a global-in-time algorithm, linear convergence.

1 Introduction

Biot's consolidation model, originally established in [9, 10], characterizes the poroelastic interaction between solid and fluid media. Governed by time-dependent partial differential equations, this model finds wide-ranging applications in fields such as geo-

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sciences, petroleum engineering, and biomedical engineering [21, 23]. Analytical solutions for this model are challenging to derive, making the numerical solutions more appealing. Several numerical methods have been developed with good success, including finite volume methods [29], virtual element methods [15], mixed finite element methods [16, 18, 22, 25, 30], space-time finite element methods [8, 24] and hybrid high-order methods [11]. Many previous studies have adopted a two-field formulation [12, 17], employing solid displacement and fluid pressure as primary variables. However, such a two-field formulation may lead to numerical difficulties [34, 40], such as elasticity locking and pressure oscillation. To address these issues, we follow the approach proposed in [26, 32] by introducing an intermediate variable referred to as "total pressure" to develop the three-field Biot's model. More clearly, we consider the three-field Biot's model as follows:

$$-2\mu\mathrm{div}(\varepsilon(\boldsymbol{u}))+\nabla\xi=\boldsymbol{f} \qquad \text{in } \Omega\times(0,T], \tag{1.1a}$$

$$\mathrm{div}\boldsymbol{u}+\frac{1}{\lambda}\xi-\frac{\alpha}{\lambda}p=0 \qquad \text{in } \Omega\times(0,T], \tag{1.1b}$$

$$\left(c_0+\frac{\alpha^2}{\lambda}\right)\partial_t p-\frac{\alpha}{\lambda}\partial_t\xi-\mathrm{div}K(\nabla p-\rho_f\boldsymbol{g})=Q_f \qquad \text{in } \Omega\times(0,T], \tag{1.1c}$$

$$\boldsymbol{u}=\boldsymbol{0} \quad \text{on } \Gamma_{\boldsymbol{u}}\times(0,T], \quad (\sigma(\boldsymbol{u})-\alpha p\boldsymbol{I})\boldsymbol{n}=\boldsymbol{h} \qquad \text{on } \Gamma_\sigma\times(0,T], \tag{1.1d}$$

$$p=0 \quad \text{on } \Gamma_p\times(0,T], \quad K(\nabla p-\rho_f\boldsymbol{g})\cdot\boldsymbol{n}=g_2 \qquad \text{on } \Gamma_q\times(0,T], \tag{1.1e}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}^0, \quad p(0)=p^0, \quad \xi(0)=\alpha p^0-\lambda\mathrm{div}\boldsymbol{u}^0 \qquad \text{in } \Omega. \tag{1.1f}$$

Here, Ω is a bounded domain in $\mathbb{R}^d$ ($d=2$ or 3) with boundary $\partial\Omega=\Gamma_{\boldsymbol{u}}\cup\Gamma_\sigma=\Gamma_p\cup\Gamma_q$ with $|\Gamma_{\boldsymbol{u}}|>0, |\Gamma_p|>0$, and $T>0$ is the final time. The primary unknowns are the displacement vector of the solid $\boldsymbol{u}$, the total pressure ξ, and the fluid pressure p. The terms $\sigma(\boldsymbol{u})=2\mu\varepsilon(\boldsymbol{u})+\lambda\mathrm{div}\boldsymbol{I}$ and $\varepsilon(\boldsymbol{u})=\frac{1}{2}[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T]$ denote the effective stress and the strain tensor, respectively, $\alpha>0$ is the Biot-Willis constant which is close to 1, $\boldsymbol{f}$ is the body force, $c_0\geq 0$ is the specific storage coefficient, K represents the hydraulic conductivity, ρ_f is the fluid density, $\boldsymbol{g}$ is the gravitational acceleration, Q_f is a source or sink term, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{n}$ is the unit outward normal to the boundary, and Lamé constants λ and μ are computed from the Young's modulus E and the Poisson ratio ν:

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu=\frac{E}{2(1+\nu)}.$$

We denote the classical Sobolev space of functions with distributional derivatives of order up to k belonging to $L^2(\Omega)$ by $H^k(\Omega)$, equipped with the norm $\|\cdot\|_{H^k(\Omega)}$. We also denote $H^k_{0,\Gamma}(\Omega)$ be the subspace of $H^k(\Omega)$ with the vanishing trace on $\Gamma\subset\partial\Omega$. Let $\boldsymbol{V}=\boldsymbol{H}^1_{0,\Gamma_{\boldsymbol{u}}}(\Omega)$, $W=L^2(\Omega)$ and $M=H^1_{0,\Gamma_p}(\Omega)$. For $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}$, $\xi,\phi\in W$, and $p,\psi\in M$, we define the following