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Spatial-Temporal Adaptive-Order Positivity-Preserving WENO Finite Difference Scheme with Relaxed CFL Condition for Euler Equations with Extreme Conditions

Jia-Le Li[1], Wai-Sun Don[1,2], Cai-Feng Wang[1] and Bao-Shan Wang[1,*]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao, Shandong 266100, China*
[2] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Kowloon, Hong Kong, China*

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Abstract. In extreme scenarios, classical high-order WENO schemes may result in non-physical states. The Positivity-Preserving Limiter (PP-Limiter) is often used to ensure positivity if CFL ≤ 0.5 with a third-order TVD Runge-Kunta (RK3) scheme. This study proposes two novel conservative WENO-Z methods: AT-PP and AO-PP to improve efficiency with $0.5 < \text{CFL} < 1$ if desired. The AT-PP method detects negative cells after each RK3 stage posteriori and computes a new solution with the PP-Limiter (CFL < 0.5) for that step. The AO-PP method progressively lowers the WENO operator's order and terminates with the first-order HLLC solver, proven positivity-preserving with CFL < 1, only at negative cells at that RK3 stage. A single numerical flux enforces conservation at neighboring interfaces. Extensive 1D and 2D shock-tube problems were conducted to illustrate the performance of AT-PP and AO-PP with CFL $= 0.9$. Both methods outperformed the classical PP-Limiter in accuracy and resolution, while AO-PP performed better computationally in some cases. The AO-PP method is globally conservative and accurate, adaptiveness, and robustness while resolving fine-scale structures in smooth regions, capturing strong shocks and gradients with ENO-property, improving computational efficiency, and preserving the positivity, all without imposing a restrictive limit on the CFL condition.

AMS subject classifications: 58J45, 35L65, 65M06

Key words: Adaptive-CFL and adaptive-order method, positivity-preserving, relaxed CFL condition, WENO, extreme problems.

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*Corresponding author.
Email: lijiale@stu.ouc.edu.cn (J.-L. Li), donwaisun@outlook.com (W.-S. Don), wangcaifeng@stu.ouc.edu.cn (C.-F. Wang), wbs@ouc.edu.cn (B.-S. Wang)

1 Introduction

High-order characteristic-wise nonlinear weighted essentially non-oscillatory (WENO) finite difference schemes are popular for solving hyperbolic conservation laws to capture shocks and strong gradients essentially non-oscillatory (ENO-property) and efficiently resolve small-scale structures. The original WENO scheme was first introduced by Liu et al. [24] and generalized by Jiang and Shu [20]. In [20], a general framework to construct arbitrary order accurate finite difference WENO-JS schemes was proposed with the new smoothness indicators and JS-type weights. The key idea behind these schemes is to use a dynamic set of substencils that adapt to a higher order (global) polynomial approximation at smooth stencils or a lower order polynomial approximation that minimizes the interpolation across discontinuities. One popular improved version of WENO-JS schemes with the Z-type weights (WENO-Z) and its higher order general framework was developed by Borges et al. [2,3]. For more information on the history, developments, and applications of classical WENO schemes, we refer to [30] and the references therein.

Although the WENO schemes achieve arbitrarily high order resolution on smooth regions and capture singular structures (e.g., shock, contact discontinuity) essentially non-oscillatory, aka, ENO-property, they face difficulties when applied directly to discretization with micro-scale sharp variations. Recently, the affine-invariant WENO-Z scheme [10, 21, 34] is designed to improve the capturing of the micro-scale sharp variation (e.g., shocklets) that might appear during the complex nonlinear wave interactions of hyperbolic conservation laws (e.g., compressible turbulence). Additionally, it has been shown [34] that the Ai-WENO scheme guarantees the ENO-property for problems with a small scaling and a nonzero translation, while the classical versions of WENO-JS and WENO-Z schemes do not. However, high-order WENO-Z schemes may still generate destabilizing Gibbs oscillations near a discontinuity when solving hyperbolic conservation laws with extreme conditions, such as a large density/pressure ratio and a near-vacuum state with extremely low density/pressure (e.g., Sedov blast-wave and shock-diffraction problems). These oscillations may lead to a non-physical state in some or all mesh cells at some or all time steps during the temporal evolution. In such scenarios, special treatments must be embedded inside high-order WENO schemes to ensure the positivity of the density and pressure. The convex set of admissible states (admissible set $\mathcal{G}$) is defined as those numerical solutions that obey the valid physical state, which is the positivity of the density and pressure. The cells with the solution not in the admissible set (negative density or pressure) are called *negative cells*. The numerical solution must belong to the admissible set to be physically valid. Equivalently speaking, the complement of the admissible set is always non-empty. We remark that the negative cell is a function of space x and time t, and the numerical solution $\mathbf{Q}(x,t)$ belongs to the admissible set $\mathcal{G}$, that is, $\mathbf{Q}(x,t)\in\mathcal{G}$. In this study, we shall consider only the seventh-order WENO-Z scheme as the base scheme. The fifth-, third-, and first-order versions usually behave much better, and no limiter is needed for most extreme problems considered.

A mature class of the *prior* positivity-preserving methods has been developed. Mo-