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Petrov-Galerkin Immersed isoP_2-P_0 Interface Method for Solving Stokes Interface Problem

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Abstract. In this paper, we develop an immersed Petrov-Galerkin finite element method for solving two-dimensional Stokes interface problems. The proposed method do not require the solution mesh to align with the fluid interface. We utilize the isoP_2-P_0 element, which adopts piecewise linear approximation for velocity on fine elements and piecewise constant approximation for pressure on coarse elements. The vector-valued solution map is constructed to approximate the velocity and pressure based on the jump conditions across the interface. Several numerical experiments demonstrate that the proposed method maintain the optimal convergence rate in the L_2-norm and the H_1-norm for the velocity and the L_2-norm for the pressure.

AMS subject classifications: 65N30, 76D07

Key words: Two-phase flow, discontinuous viscosity, immersed FEM.

1 Introduction

Multi-phase flow interface problems are caused by lots of medium and physical interaction, which is widely used in practical life. For instance, simulating the blood flow in human heart [8, 16], describing the elastic interface problem of various material behaviors [15], Darcy-Stokes coupling model [17], moving interface problem [2], fluid-solid coupling problem and so on. The two-phase incompressible flow is the main motivation for considering the Stokes interface problem. Problems like that are generally modeled by the Stokes equations with discontinuous viscosity coefficients.

Many scholars have developed many numerical methods to solve interface problems. According to the relationship between the grid and the interface, two types of finite element methods are developed, one is the interface-fitted method and the other is interface-unfitted method. The interface-fitted method can be directly applied and obtain optimal

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error estimation. However, when dealing with the complex interface or moving interface problem, the computational cost of meshing will be quite huge.

Interface-unfitted method is independent of the interface and allows the intersection of the element and the interface. Due to the discontinuities in the pressure and the derivatives of velocity across the interface, the interface-unfitted numerical methods with optimal convergence rates is hard to develop. In previous works, there are several kinds of interface-unfitted grid methods. One is the extended finite element method (XFEM) which is very sensitive to the location of the interface. Based on this, Hansbo and Hansbo combined the Nitsche method and XFEM (also called cut-FEM or Nitsche-XFEM) to solve elliptic interface problems in [4]. The technique of Nitsche-XFEM method is to construct independent solutions on each separated subregion and to use the Nitsche technique to limit the jump conditions to the interface. By selecting local jump coefficients on each element, uniformly optimal convergence order is obtained. See [5, 10] and references therein.

Another type of interface-unfitted method are the immersed finite element method [11, 12, 14]. This method modifies the traditional interface element basis according to the interface conditions while maintaining the degree of freedom. Li proposed a piecewise linear immersed finite element method [13] for one-dimensional elliptic interface problem, and then Adjerid and Lin extended this method to high-order approximation [3].

Adjerid, Chaabane and Lin first proposed an immersed finite element method for solving the Stokes interface problem in [1]. The immersed Q_1-Q_0 element is constructed to satisfy the jump conditions in this paper. Then Adjerid et al. applied this methodology for the Stokes moving interface problem in [2]. Zhang and Jones developed immersed CR-P_0 and rotated Q_1-Q_0 finite element methods in [9]. However, for the immersed interface finite element method, its basis function construction depends on the interface jump condition, which makes error analysis very difficult.

In this paper, we develop an immersed Petrov-Galerkin interface method for Stokes interface problem. Our method is based on the conforming framework inspired by [6,7]. The proposed method utilizes the isoP_2-P_0 element, which adopts piecewise linear approximation for velocity on small elements and piecewise constant approximation for pressure on large elements. Interface is approximated by the line segments which is calculated by the level-set function. The trial solution maps are constructed according to the jump conditions, while the test functions take the general isoP_2-P_0 element space. In addition, this method does not change the original degree of freedom and the position of the degree of freedom, that is, the method is structure-preserving and is very beneficial to deal with the problem of moving interface. The proposed method only needs to construct the linear estimation of the velocity once on the interface cell, which is easier to implement than the general immersed method. Numerical experiments show that the proposed method can obtain optimal converge rate.

The outline of the paper is as follows. In Section 2, we introduce the model of the Stokes interface problems and give the weak formulation. We develop the Petrov-Galerkin immersed interface finite element method in Sections 3. In Section 4, some numerical ex-