Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **17**, No. 3, pp. 956-988

DOI: 10.4208/aamm.OA-2022-0320
June 2025

A Robust Three-Level Time Split High-Order Leapfrog/ Crank-Nicolson Scheme for Two-Dimensional Sobolev and Regularized Long Wave Equations Arising in Fluid Mechanics

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Received 5 December 2022; Accepted (in revised version) 6 June 2023

Abstract. This paper develops a robust three-level time split high-order Leapfrog/ Crank-Nicolson technique for solving the two-dimensional unsteady Sobolev and regularized long wave equations arising in fluid mechanics. A deep analysis of the stability and error estimates of the proposed approach is considered using the $L^\infty(0,T;H^2)$-norm. Under a suitable time step requirement, the theoretical studies indicate that the constructed numerical scheme is strongly stable (in the sense of $L^\infty(0,T;H^2)$-norm), temporal second-order accurate and convergence of order $\mathcal{O}(h^{8/3})$ in space, where h denotes the grid step. This result suggests that the proposed algorithm is less time consuming, faster and more efficient than a broad range of numerical methods widely discussed in the literature for the considered problem. Numerical experiments confirm the theory and demonstrate the efficiency and utility of the three-level time split high-order formulation.

AMS subject classifications: 65M12, 65M06
Key words: Sobolev and regularized long wave equations, Leapfrog scheme, Crank-Nicolson method, three-level time-split high-order Leapfrog/Crank-Nicolson approach, stability analysis, error estimates.

1 Introduction

The exploration of analytical solutions of the time-dependent partial differential equations (PDEs) plays a vital role in describing the behavior of different physical and biological phenomena arising in the areas of mathematical biology, fluid dynamics, engineering, chemical theory, bio-modeling and fluid mechanics [16,20,25,26,28,29,34,41,44,47].

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A large class of unsteady systems of PDEs have been used to model various problems in chemistry, physics, biology and engineering such as: chemical kinematics, fluid mechanics, electricity, nonstationary process in semiconductors in the presence of sources, propagation of wave and shallow water waves, optical fibers, flow of heat, plasma physics, immunology, quantum mechanics, Sobolev and regularized long wave models and biology [1,2,6,8–10,18,19,23,24,33,35,43,45,48,49] and references therein. The evolutionary two-dimensional Sobolev and regularized long wave problems usually arise in the flow of fluids to explaining the motion of wave in media. This model is associated with the Rossy and drift waves in rotating fluids and plasmas, respectively, and it describes a broad range of applications in different branches in engineering and science [21,43]. Developing both exact and efficient numerical solutions for different types of Sobolev and regularized long wave equations is an attractive area of research in applied science.

In this paper, we consider the two-dimensional Sobolev and regularized long wave equations defined in [43] as

$$u_t-\alpha\Delta u_t-\gamma\Delta u+(\beta,\beta)\cdot\nabla u=f(x,y,t,u,u_x,u_y) \quad \text{on } \Omega\times(0,T], \tag{1.1}$$

with initial condition

$$u(x,y,0)=u_0(x,y) \quad \text{on } \Omega\cup\partial\Omega, \tag{1.2}$$

and boundary condition

$$u(x,y,t)=g(x,y,t) \quad \text{on } \partial\Omega\times[0,T], \tag{1.3}$$

where $f(x,y,t,u,u_x,u_y)=f_1(x,y,t,u,u_x)+f_2(x,y,t,u,u_y)$, and f_m, for $m=1,2$, are nonlinear functions. For the sake of stability analysis and error estimates, we assume that the functions f_m are locally Lipschitz with respect to the unknown u. Δ and ∇ designate the Laplacian and gradient operators, respectively, (β,β) denotes the velocity vector, u_z means $\frac{\partial u}{\partial z}$, for $z=x,y,t$. α, β and γ are nonnegative constant less than one, with $\alpha\neq 0$, u_0 and g represent the initial and boundary conditions, respectively. Eq. (1.1) is a third-order mixed derivative in both time and space and it is referred to the Sobolev equations arising in flow of liquid through the theory of heat conduction, fissured rocks and nonsteady flow [2, 5, 46]. When $\alpha=0$, Eq. (1.1) becomes a nonlinear convection-diffusion-reaction model which has been widely studied in the literature [26,27,30–32,36–40,42]. For $\alpha\neq 0$, a large set of numerical methods have been developed in an approximate solution of the initial-boundary value problem (1.1)-(1.3), such as: Galerkin finite element methods, split least-square plan, Runge Kutta method, conservative scheme, a computational approach, Lumped Galerkin procedure. For more details, we refer the readers to [3,4,7,8,11–15,51] and references therein. For some methods listed above, either the stability analysis or the error estimates has not been considered. In this work, we develop a three-level time split high-order Leapfrog/Crank-Nicolson approach for solving the partial differential equation (1.1) subjects to initial-boundary conditions (1.2)-(1.3). Under an appropriate time step limitation, the proposed formulation is strongly stable (in the